More Exact Matching

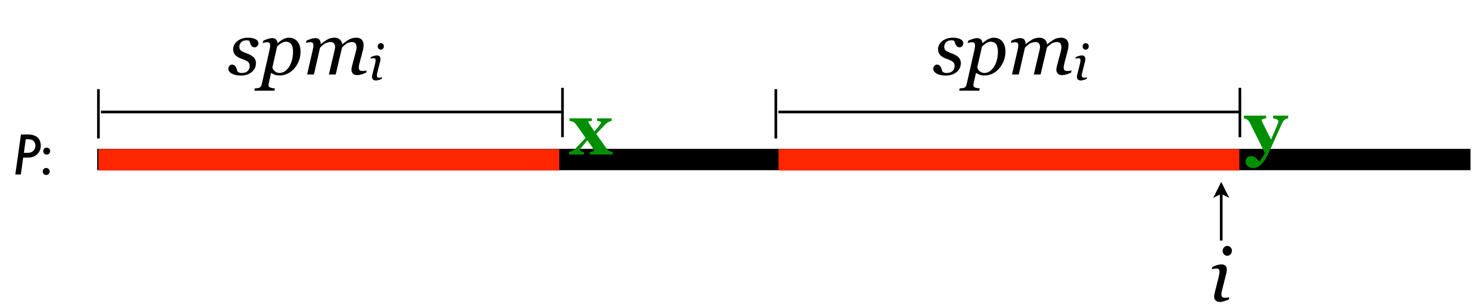
(Following Gusfield Chapter 2)

Knuth-Morris-Pratt

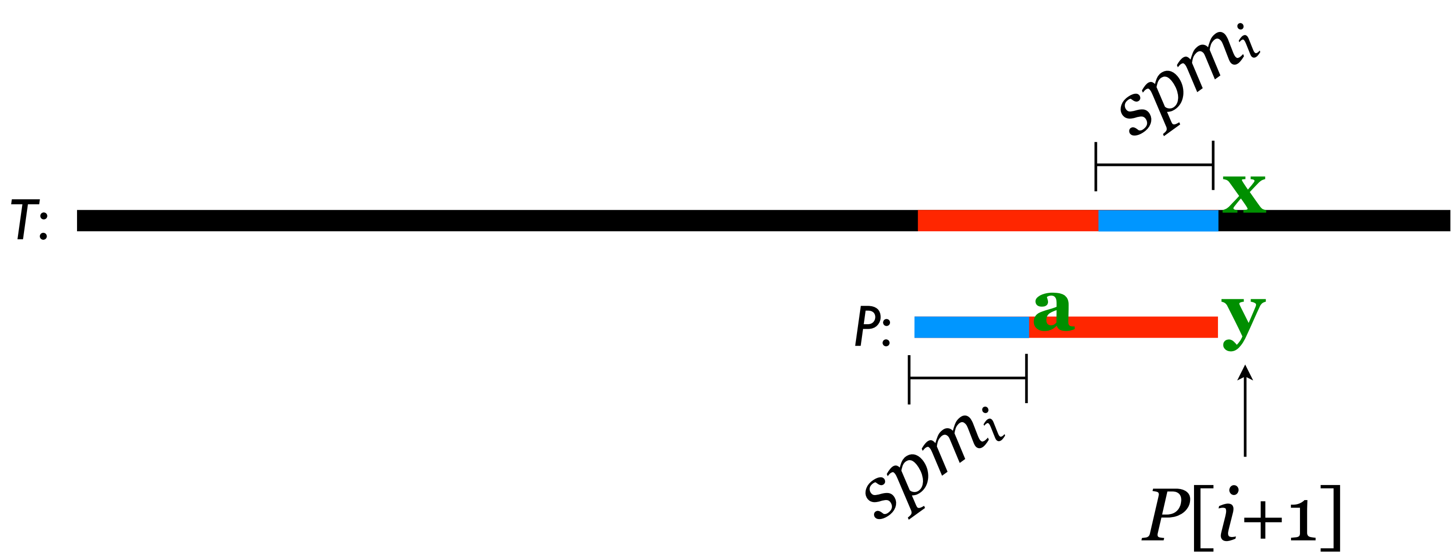
Knuth-Morris-Pratt (KMP)

- Shift by more than 1 place, if possible, upon mismatch.

Def. $spm_i(P)$ = the length of the longest substring of P that *ends* at i > 1 and matches a prefix of P **and** such that $P[i+1] \neq P[spm_i + 1]$.
("*spm*" stands for suffix, prefix, mismatch.)

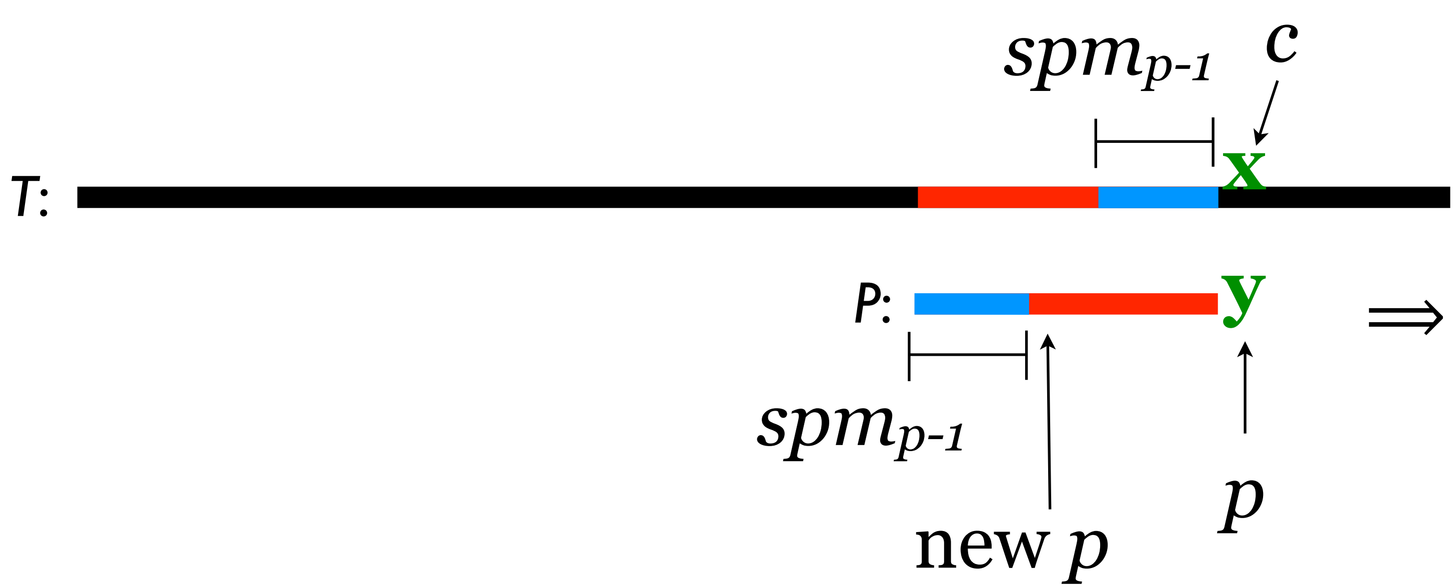


KMP Algorithm: Suppose mismatch at i+1 of P:



$\Rightarrow$ can shift by: $i - spm_i$

KMP



$\Rightarrow$ can set new p to $spm_{p\text{-}1}$ +1

```
c = p = 1     // ptrs into T and P, respectively
while c ≤ |T| - |P| + p:
  while P[p] = T[c] and p ≤ n: // compare P and T
    p++
    c++
  if p = n + 1: print "Found at", c - n   // if found
  if p = 1: // failure at start means inc c
    c++
  else:
    p = spm_{p-1} + 1   // "shift" by n - spm_{p-1} (even if p=n+1)
```

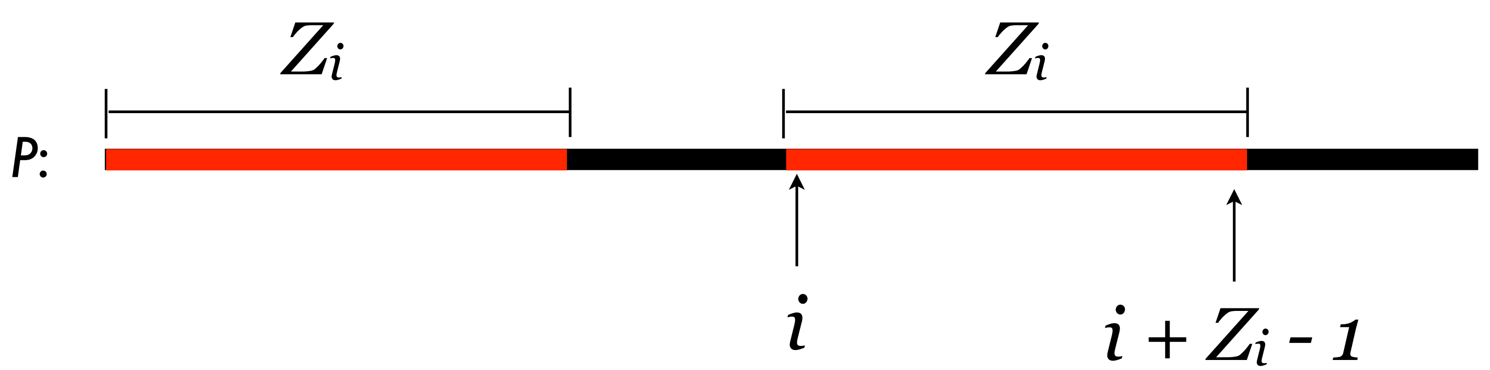
KMP Running Time

Pseudocode runs in O($|T|$) time (making at most $2|T|$ comparisons):

- In each iteration of the outer **while** loop, at most one character is compared that was compared in a previous iteration.
- Total comparisons: $\leq |T| + s$, where s = # of times through the outer while loop.
- $s \leq |T|$ since P is shifted by ≥ 1 each time.
- Therefore: O($|T|$) for the pseudocode on previous page.

Recall: Fundamental Preprocessing

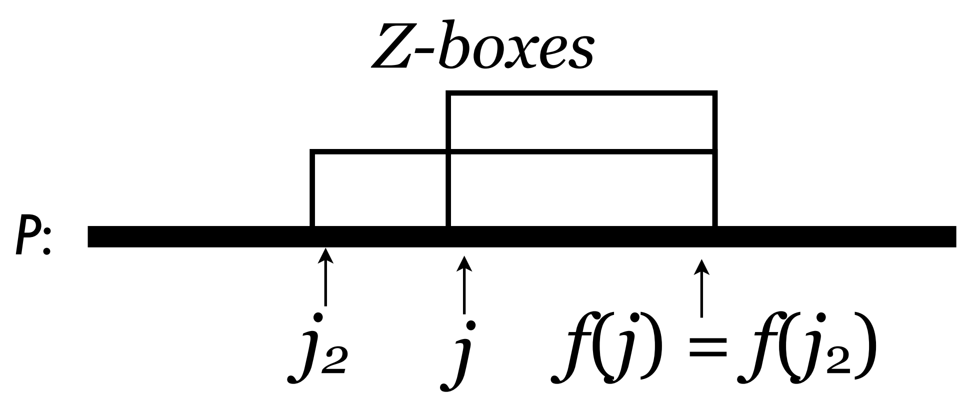
Def. $Z_i(P)$ = the length of the longest substring of P that starts at $i > 1$ and matches a prefix of P.



- P = "aardvark": $Z_2 = 1$, $Z_6 = 1$
- P = "alfalfa": $Z_4 = 4$
- P = "photophosphorescent": $Z_6 = Z_{10} = 3$

Computing spm_i for KMP

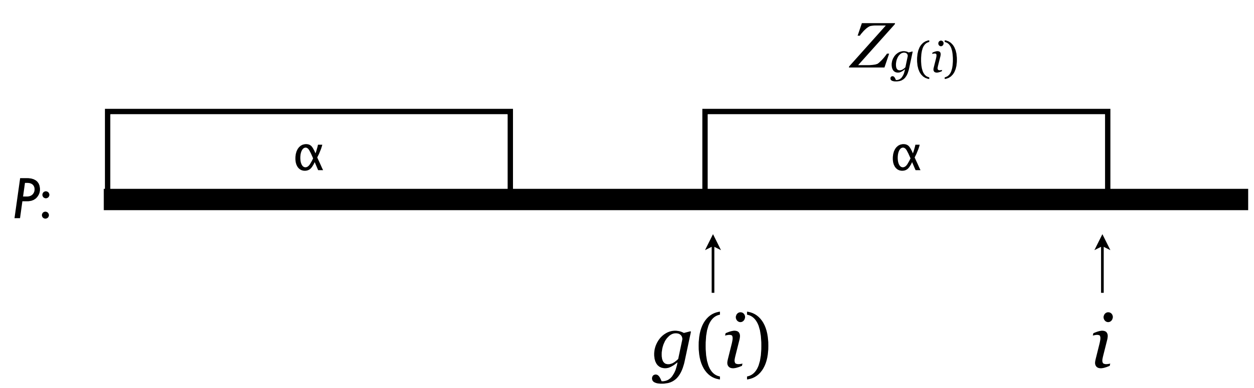
$f(j)$ = the right end of the Z-box (if any) that starts at j.



$g(i) = \min\{j : f(j) = i\}$ or 0 if empty set.

Thm. $spm_i = Z_{g(i)}$ if $g(i) > 0$ otherwise 0

Proof.



$P[g(i)..i] = P[1..Z_{g(i)}]$ by the definition of Z.

Also, $P(i+1) \neq P[Z_{g(i)}+1]$, otherwise $Z_{g(i)}$ would be bigger.

So, $spm_i \geq Z_{g(i)}$. But it can't be longer, because otherwise $g(i)$ would be smaller.

Boyer-Moore

Boyer-Moore Main Ideas

- For a given shift, compare P to T from *right to left*.

```
thequickbrownfox
        x||||
        crown
```

- Two rules for shifting:

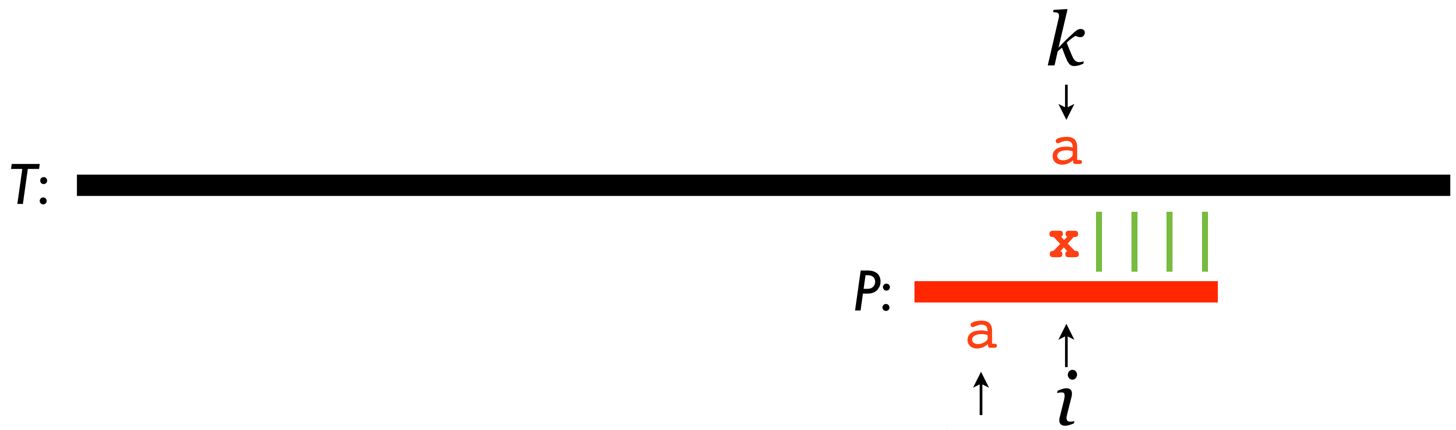
  (1) Bad Character Rule

  (2) Good Suffix Rule

Bad Character Rule

Def. $R_i(x)$ = position of the rightmost occurrence of character x before position i.

- When a mismatch occurs at pattern position i:



$$R_i(a) = R_i(T[k])$$

shift by $i - R_i(T[k])$ characters so that the next occurrence of $T[k]$ in the pattern is underneath position k in T.

(Called the “bad character rule” because it fires on a mismatch, but really it shifts so that the next *good* character matches.)

Computing $R_i(x)$

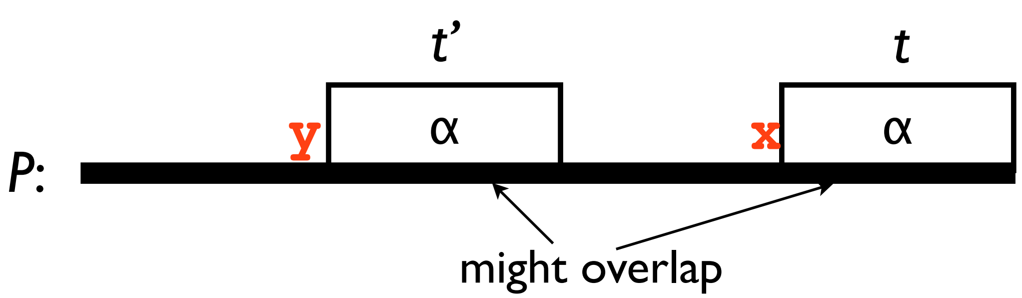
Def. $R_i(x)$ = position of the rightmost occurrence of character x before position i.

- Array $R[i,x]$ would depend on the size of the alphabet, which is undesirable.
- Better to use a collection of lists:
  - Occur[x] = positions where x occurs in P in decreasing order.
- To find $R_i(x)$:
  - scan down list x until you find first index $< i$
- Time: at most $O(n - i)$ time, since if mismatch occurred at position i then there can be at most $n - i$ items on the list that are $\geq i$.
- Only call this routine after *matching* $O(n - i)$ characters, so at most doubles the running time.

Good Shift Rule

Apply these cases in order:

Case (A):



Shift so that the rightmost occurrence of matched suffix with different preceding character is aligned to matched part of *T*.

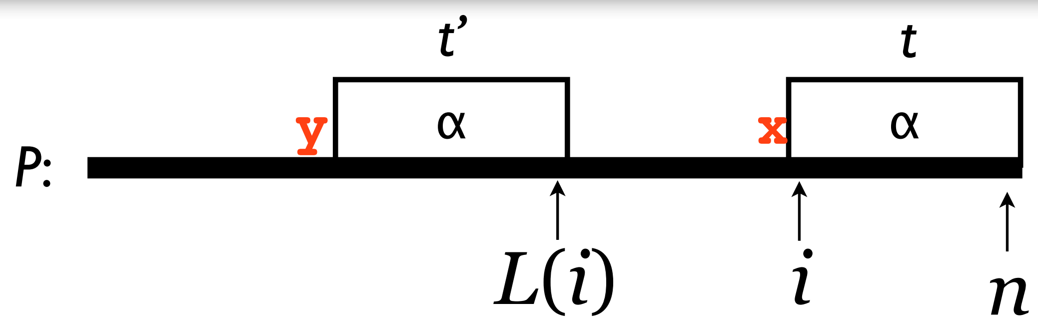
Case (B):

β longest proper prefix of *P* that matches a suffix of α. Shift so that the prefix β matches the suffix β that was matched to *T*.

Case (C): If not (A) or (B), shift |*P*| places.

Processing the good suffix rule

Def. $L(i)$ = largest index such that $P[i..n]$ matches suffix of $P[1..L(i)]$ and $P[i\text{-}1] \neq$ the character preceding that suffix (0 if no such index exists).

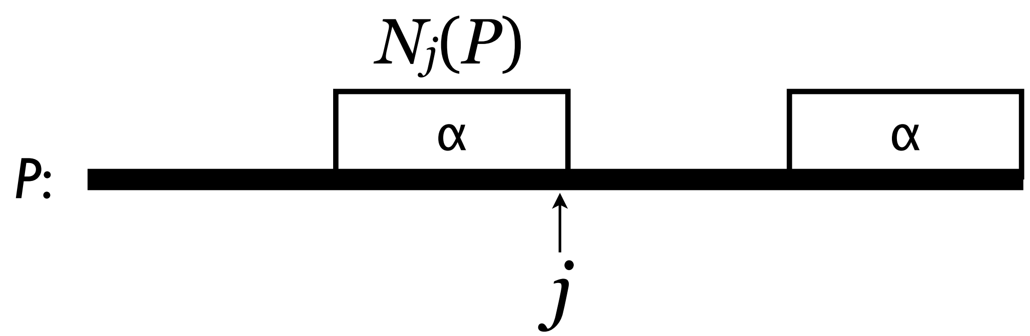


Def. $l(i)$ = size of largest suffix of $P[i..n]$ that equals some *prefix* of P (0 if none exists).

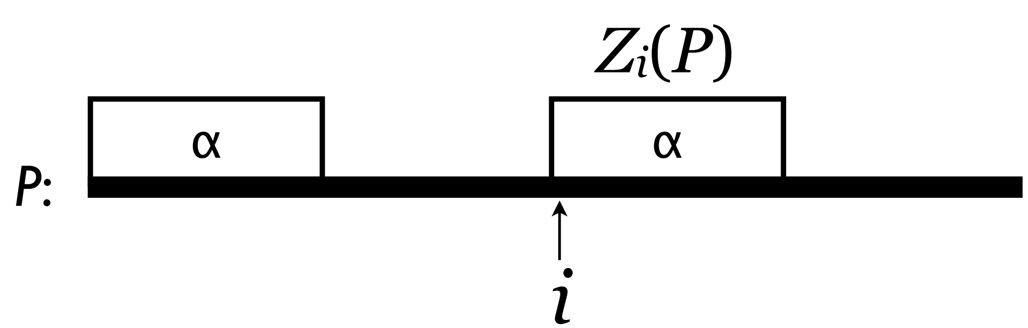
- Case (A): shift by n - $L(i)$.
- Case (B): if $L(i)$ = 0: shift by n - $l(i)$ places.
- If match: shift by n - $l(2)$ places.

Computing L(i)

Def. $N_j(P)$ = length of longest suffix of $P[1..j]$ that is also a suffix of P.



Recall: **Def.** $Z_i(P)$ = the length of the longest substring of P that starts at $i > 1$ and matches a prefix of P.

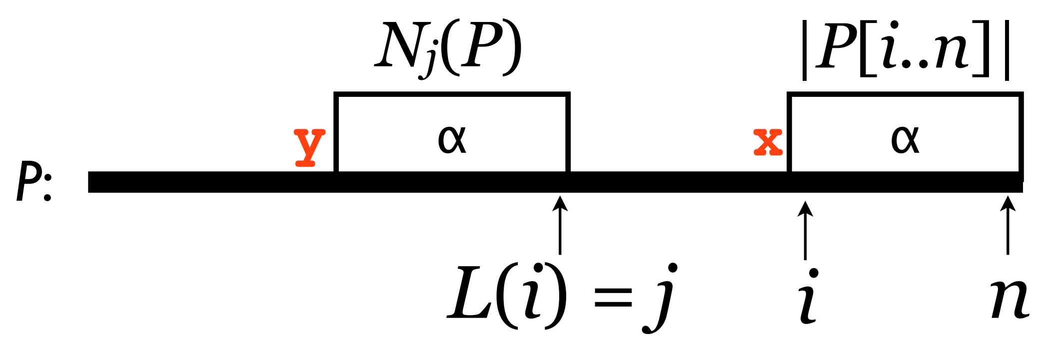


$N_j(P)$ and $Z_i(P)$ are reverses of each other:
$N_j(P) = Z_{n-j+1}(P^r)$, where P^r is P reversed.

← *Can compute in $O(n)$ time using Z-algorithm on P^r.*

Computing L(i), continued

- $L(i)$ = largest index j such that $P[i..n]$ matches suffix of $P[1..L(i)]$ and $P[i\text{-}1] \neq$ the character preceding that suffix.



- $N_j(P)$ = length of longest suffix of $P[1..j]$ that is also a suffix of P.

$\Longrightarrow$ $L(i)$ = largest index j such that $N_j(P) = |P[i..n]| = n - i + 1$

- x ≠ y because otherwise $N_j(P)$ would be longer.

```
Compute Nj[P] via Z-Algorithm for all j.
Initialize L[i] = 0 for all i.
for j = 1 to n - 1:
  i = n - Nj[P] + 1
  L[i] = j
```

Boyer-Moore

```
k = 1
while k < |T| - |P| + 1:
  Compare P to T[k..|P|] from right to left.
  s = max { bad character rule, good suffix rule, 1 }
  k += s
```

- Worst case running time = O(nm) since might shift by 1 every time.
- Despite this, Boyer-Moore often the best choice in practice because on real texts the running time is often sublinear (since the heuristics allow skipping a lot of characters).
- Extensions exist that guarantee O($|P| + |T|$) running time.