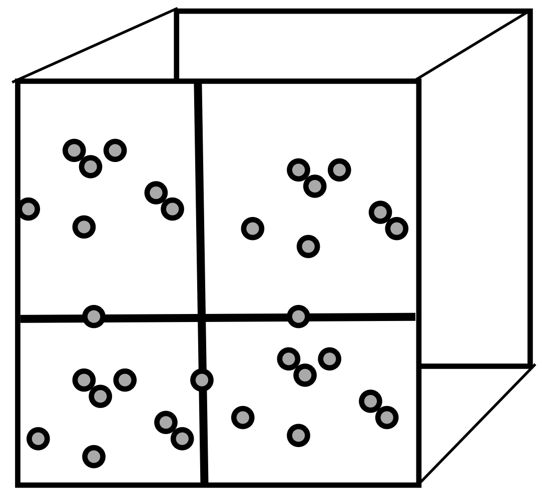
kd-Trees Continued

Generalized, incremental NN, range searching, kd-tree variants

kd-tree Variants

- How do you pick the cutting *dimension*?
  - kd-trees cycle through them, but may be better to pick a different dimension
  - e.g. Suppose your 3d-data points all have same Z-coordinate in a give region:



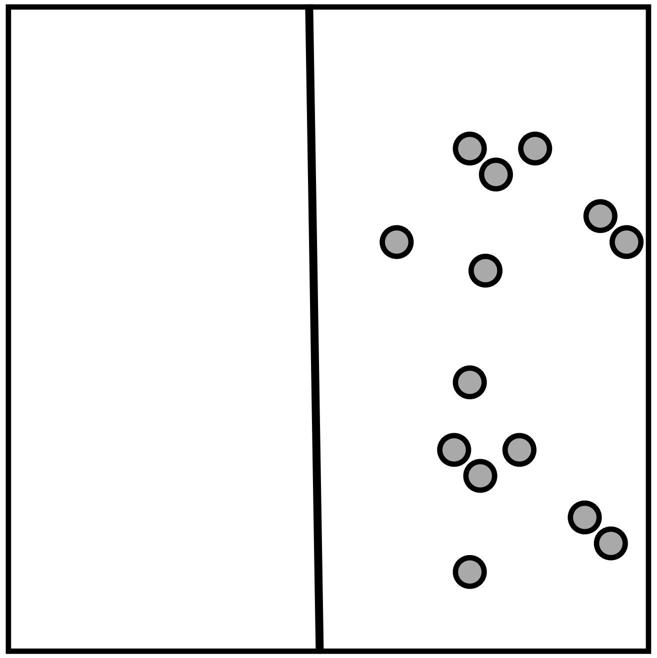
- How do you pick the cutting *value?*
  - kd-trees pick a key value to be the cutting value, based on the order of insertion
  - *optimal kd-trees:* pick the key-value as the median
  - Don't need to use key values => like PR Quadtrees => PR kd-trees

- What is the size of leaves?
  - if you allow more than 1 key in a cell: *bucket kd-trees*
- kd-trees: discriminator = (hyper)plane;
  quadtrees (and higher dim) discriminator complexity grows with *d*

Sliding Midpoint kd-trees

- PR kd-tree: split in the midpoint, along the current cutting dimension
- May result in *trivial* splits: if all points lie to one side of the median
- **Solution**: if you get a trivial split, *slide* the split so that it cuts off at least one point:

Sliding midpoint kd-tree

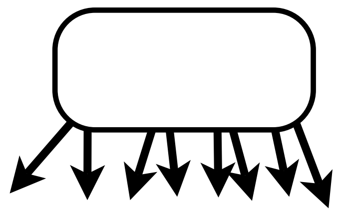


Avoids empty cells

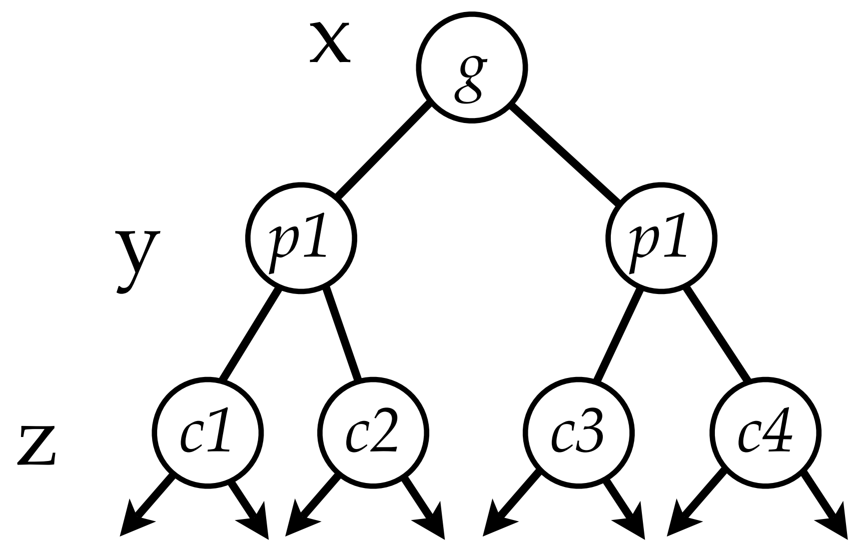
Tends to put boundaries around bounding boxes of clusters of points

kd-Trees vs. Quadtrees, another view

Consider a 3-d data set



Octtree



kd-tree

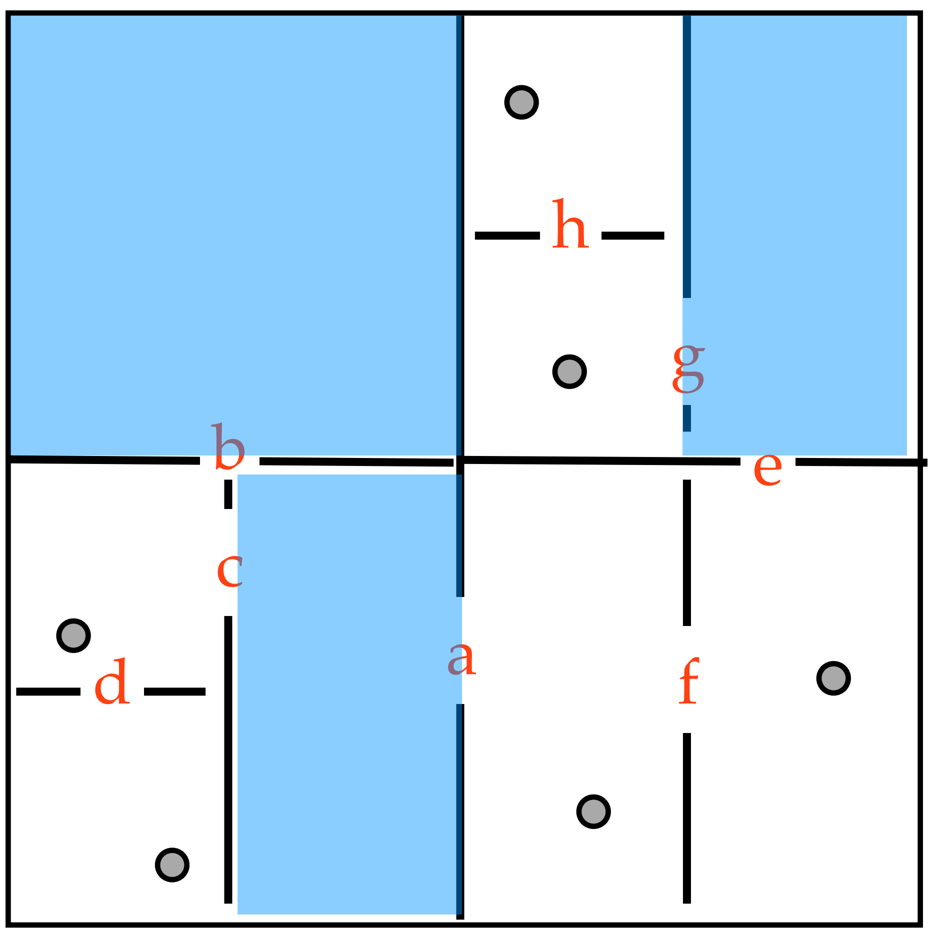
kd-tree splits the decision up over d levels

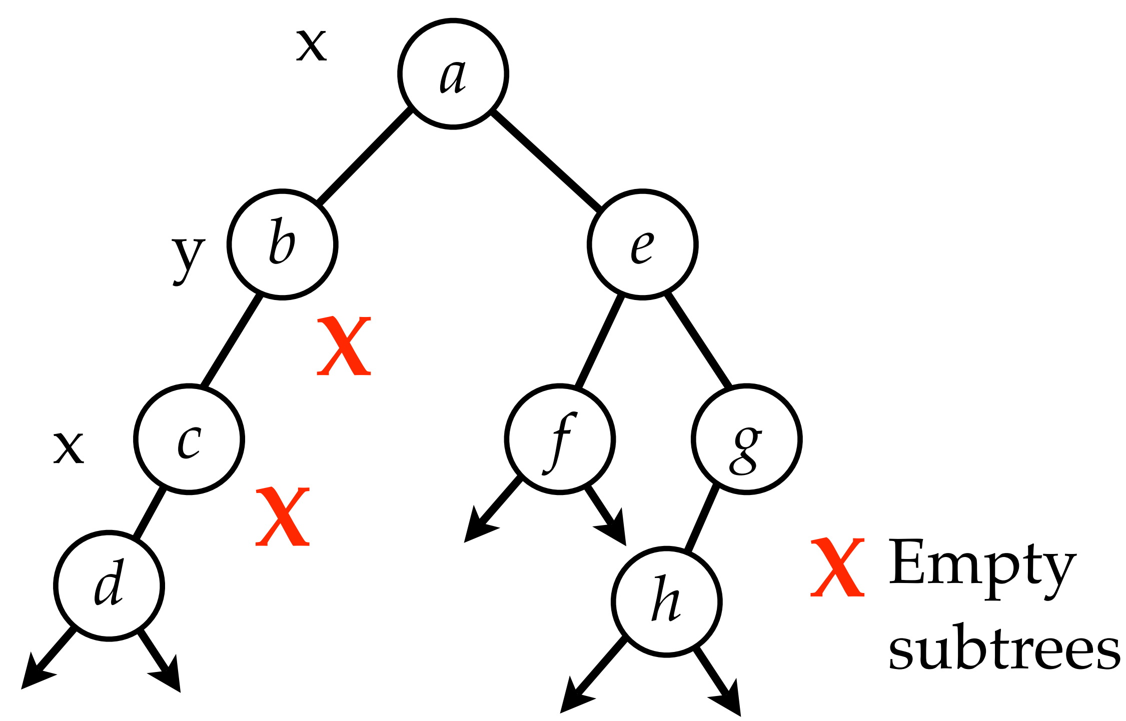
- don't have to represent levels (pointers) that you don't need

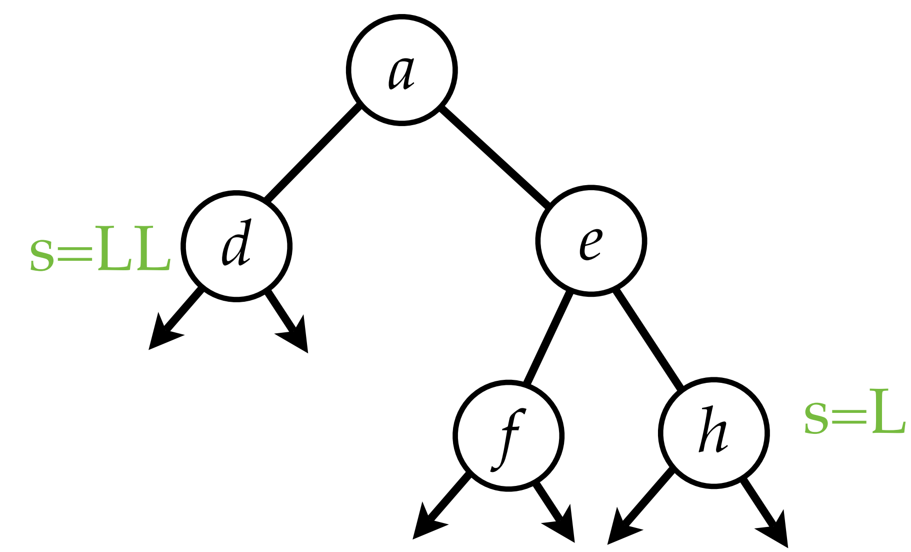
Quadtrees: one *point* determines all splits
kd-trees: flexibility in how splits are chosen

Path-compressed PR kd-trees

Empty regions







Strings of Ls and Rs tell the decisions skipped that would lead to this node

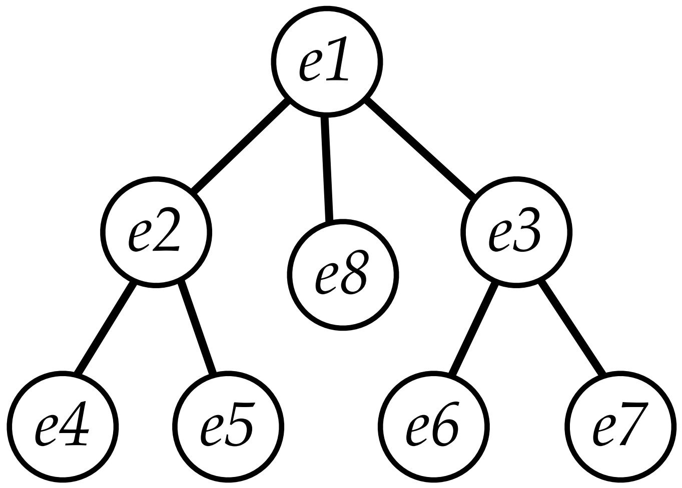
Path compressed PR kd-tree

Generalized Nearest Neighbor Search

- Saw last time: nearest neighbor search in kd-trees.
- What if you want the k-nearest neighbors?
- What if you don't know k?
  - E.g.: Find me the closest gas station with price < \$3.25 / gallon.
  - Approach: go through points (gas stations) in order of distance from me until I find one that meets the \$ criteria
- Need a NN search that will find points in order of their distance from a query point q.
- Same idea as the kd-tree NN search, just more general

Generalized NN Search

- A feature of all spatial DS we've seen so far: decompose space hierarchically.
  No matter what the DS, we get something like this:



Let the items in the hierarchy be e1,e2,e3...

Items may represent points, or bounding boxes, or ...

Let Type(e) be an abstract "type" of the object:

  we use the type to determine which distance function to use

  E.g: if Type = "bounding box" then we'd use the point-to-rectangle distance function.

A concrete example: in a Quadtree: internal nodes have type "bounding box"

Leaves would have type "point"

Generalized, Incremental NN

Let IsLeaf(), Children(), and Type() represent the decomposition tree

Let $d_t(q,e_t)$ be the distance function appropriate to compare points with elements of type t.

Idea: keep a priority queue that contains *all elements* visited so far (points, bounding boxes)

Priority queue (heap) is ordered by distance to the query point

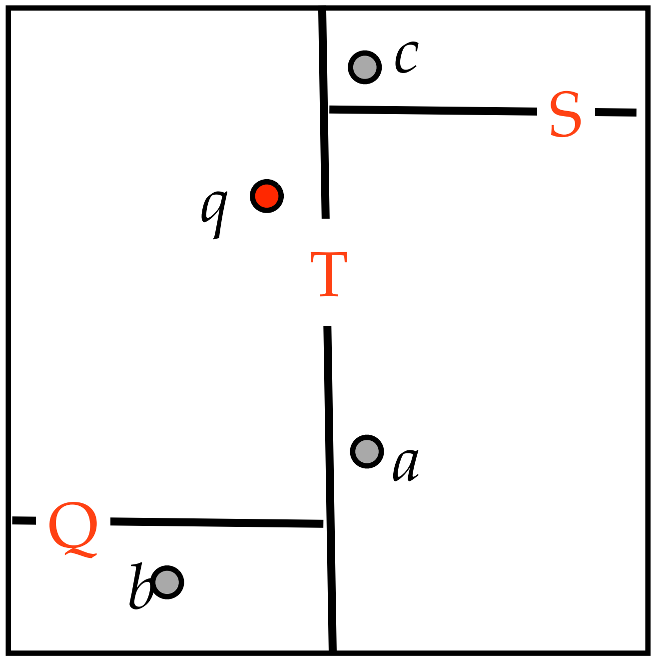
When you dequeue a point (leaf), it will be the next closest

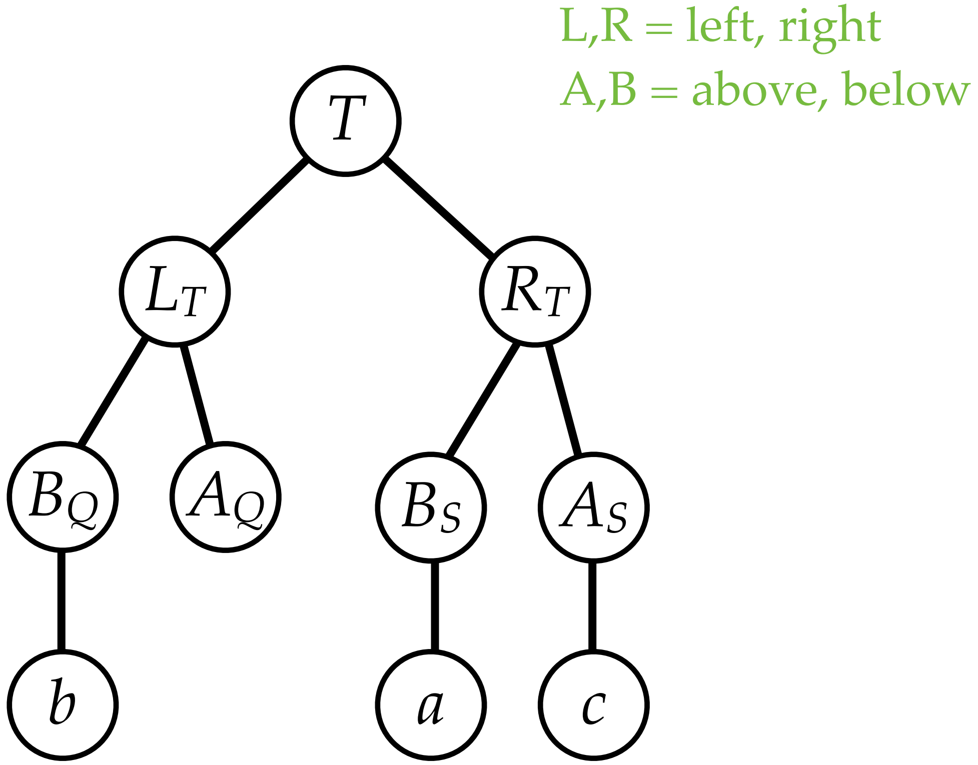
```
HeapInsert(H, root, 0)
while not Empty(H):
   e := ExtractMin(H)
   if IsLeaf(e):
      output e as next nearest
   else
      foreach c in Children(e):
        t = Type(c)
        HeapInsert(H, c, d_t(q,c))
```

$d_t(q,c)$ may be the distance to the bounding box represented by *c*, e.g.

Incremental, Generalized NN Example

Some spatial data structure:



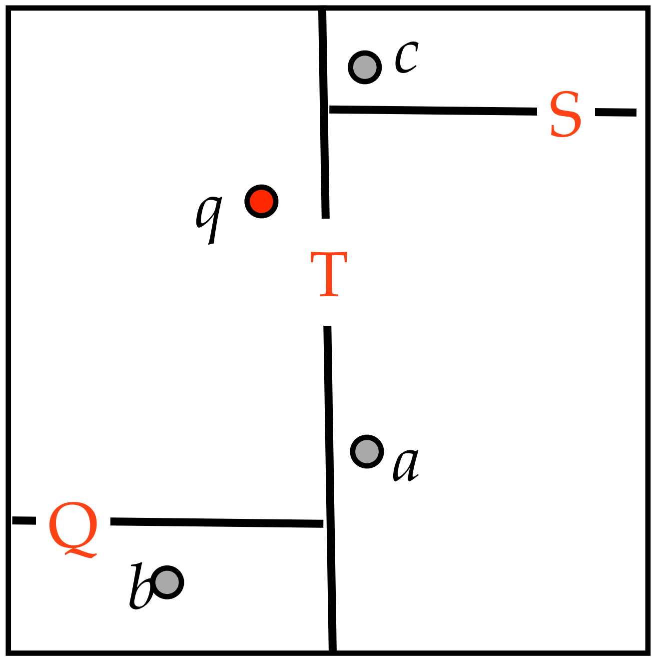


It's spatial decomposition (**NOT** the actual data structure)

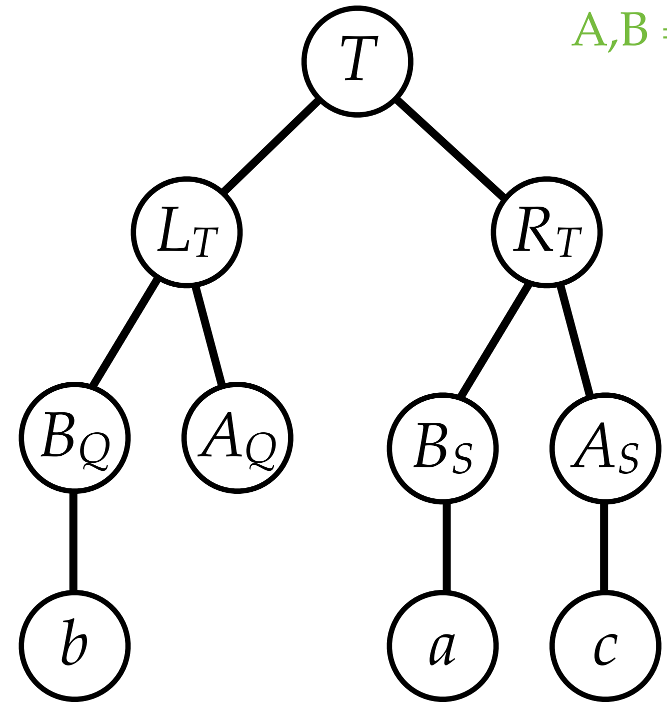
```
HeapInsert(H, root, 0)
while not Empty(H):
   e := ExtractMin(H)
   if IsLeaf(e):
      output e as next nearest
   else
      foreach c in Children(e):
        t = Type(c)
        HeapInsert(H, c, dt(q,c))
```

Incremental, Generalized NN Example

Some spatial data structure:



L,R = left, right
A,B = above, below



Its spatial decomposition (**NOT** the actual data structure)

```
HeapInsert(H, root, 0)
while not Empty(H):
   e := ExtractMin(H)
   if IsLeaf(e) && IsPoint(e):
      output e as next nearest
   else
      foreach c in Children(e):
        t = Type(c)
        HeapInsert(H, c, dt(q,c))
```

$H = []$

$H = [T]$

$H = [L_T\ R_T]$

$H = [A_Q\ R_T\ B_Q]$

$H = [R_T\ B_Q]$

$H = [B_S\ A_S\ B_Q]$

$H = [A_S\ a\ B_Q]$

$H = [c\ a\ B_Q]$

$H = [c\ a\ b]$

$H = [a\ b]$

$H = [b]$

$H = []$

Range Searching

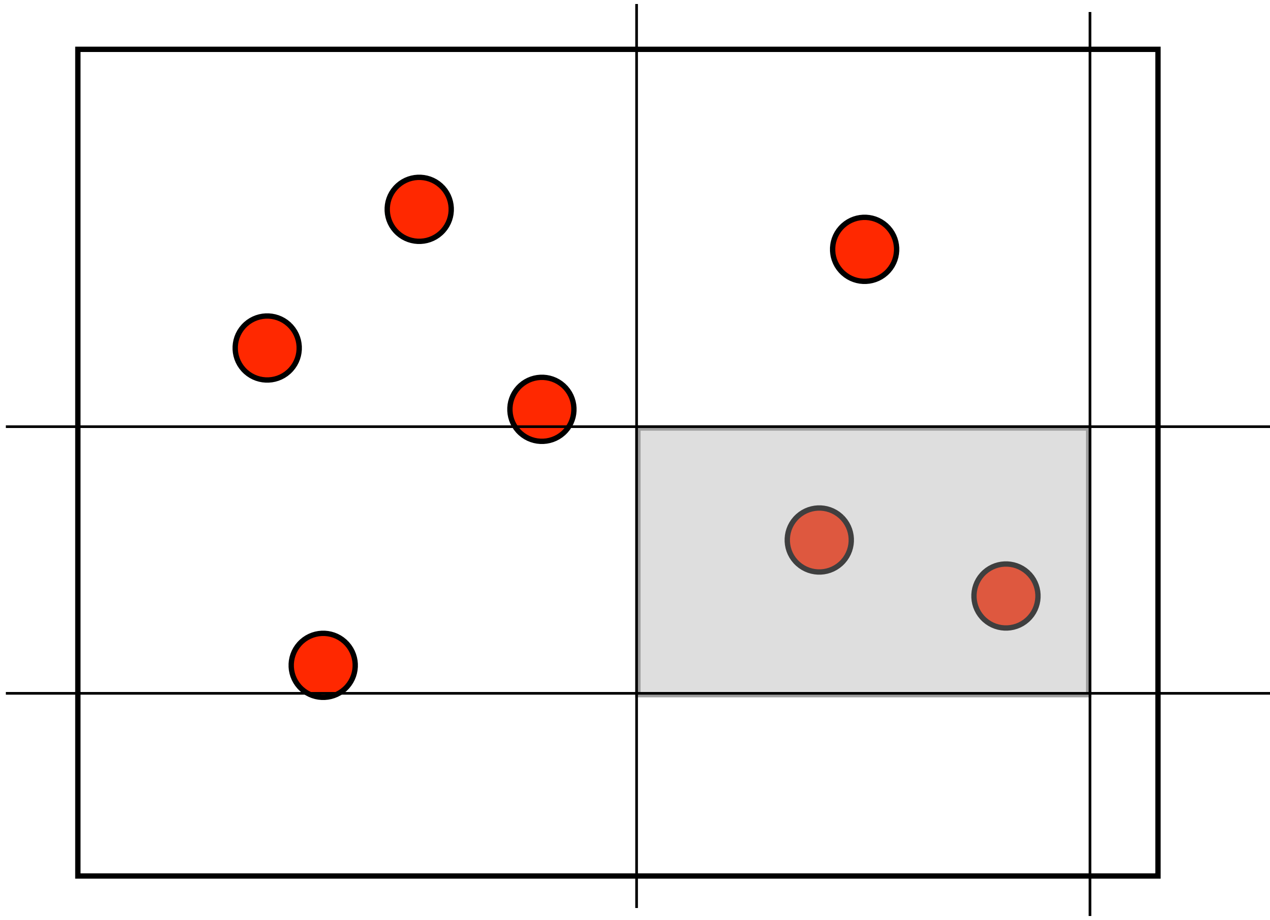
CMSC 420

Range Searching in kd-trees

- Range Searches: another extremely common type of query.

- Orthogonal range queries:
  - Given axis-aligned rectangle
  - Return (or count) all the points inside it

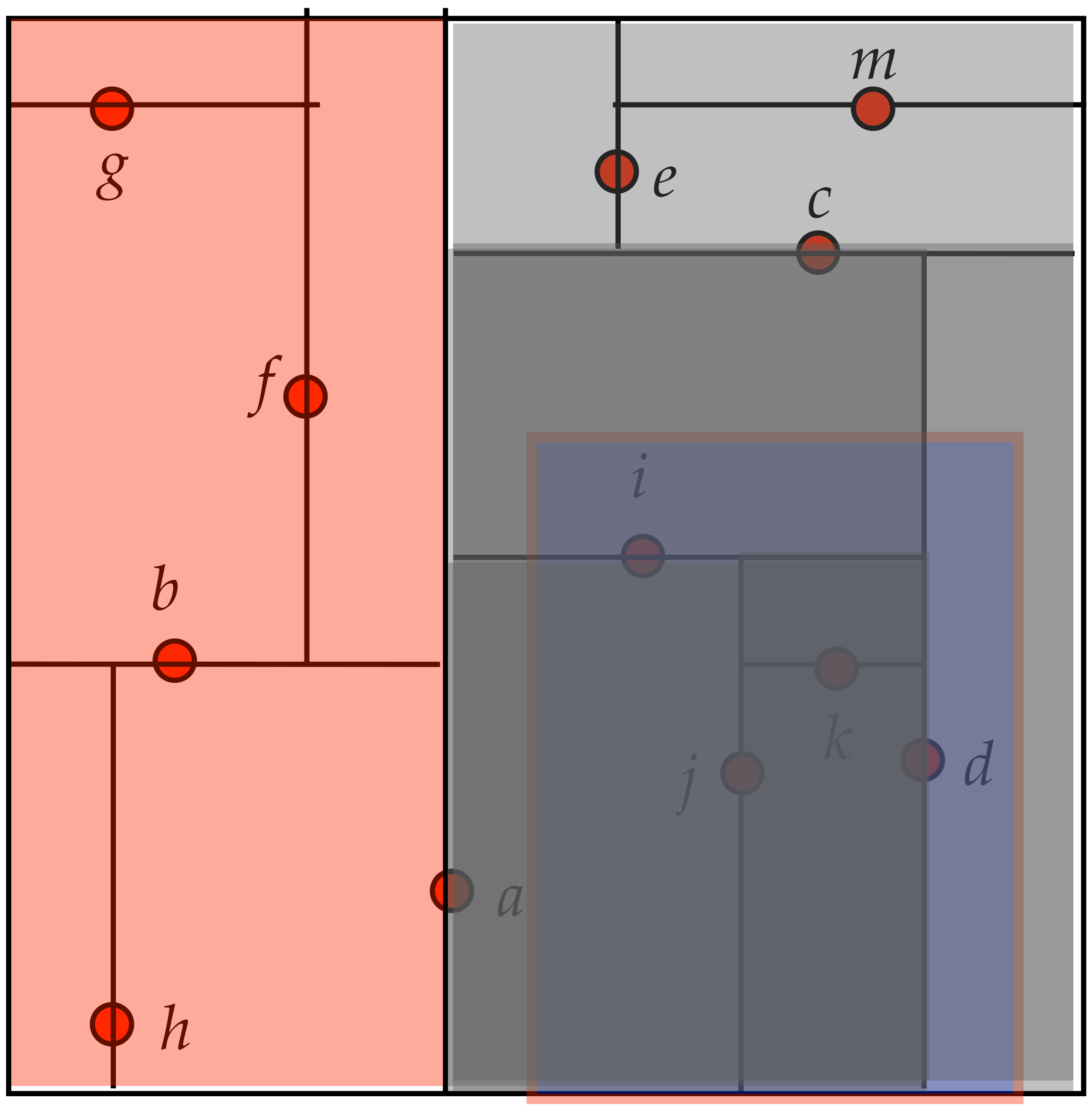
- **Example**: find all people between 20 and 30 years old who are between 5′8″ and 6′ tall.

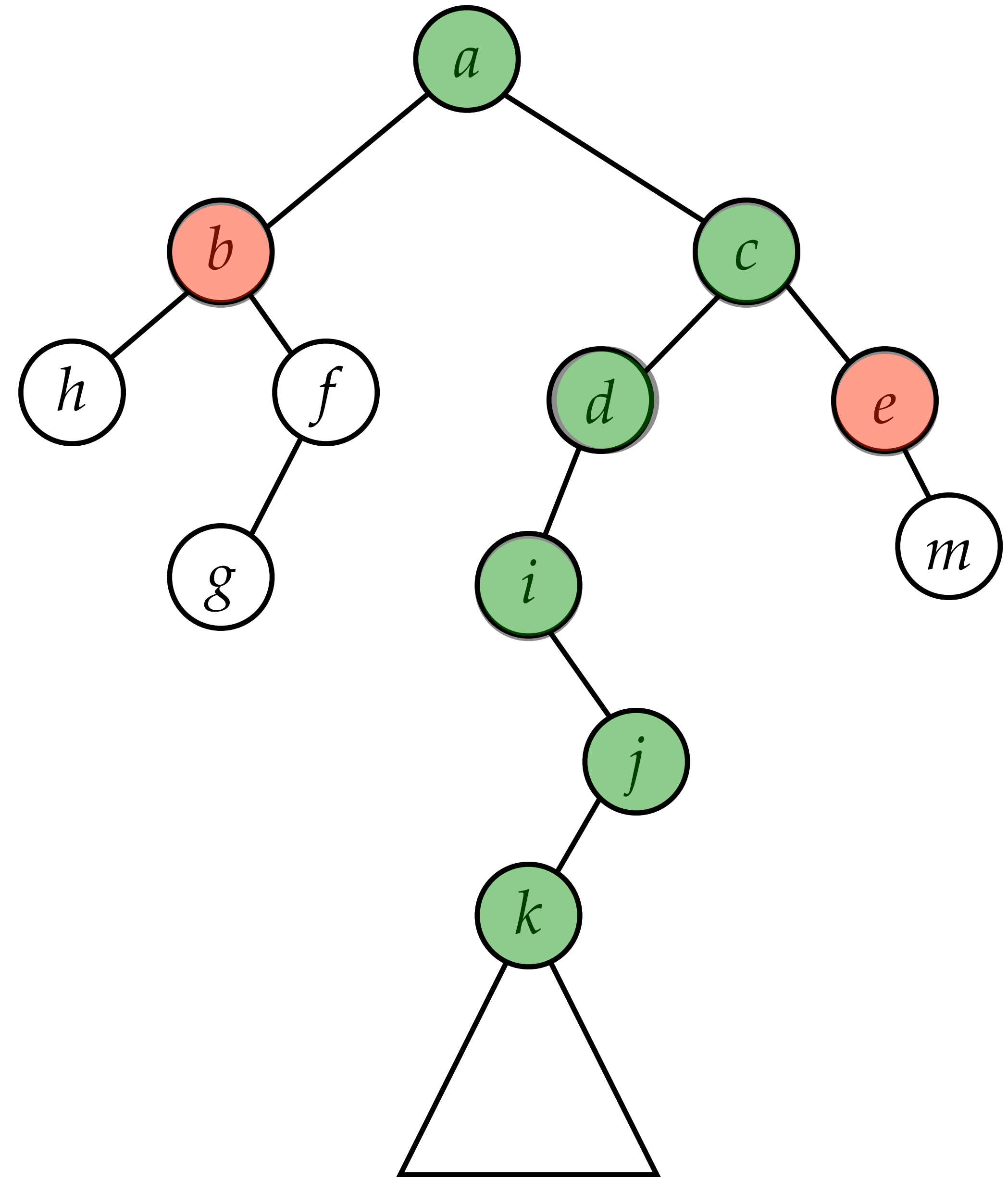


Range Searching in kd-trees

- Basic algorithmic idea:
  - traverse the whole tree, **BUT**
    - prune if bounding box doesn’t intersect with Query
    - stop recursing or print all points in subtree if bounding box is entirely inside Query

Range Searching Example





If query box doesn't overlap bounding box, stop recursion

If bounding box is a subset of query box, report all the points in current subtree

If bounding box overlaps query box, recurse left and right.

Range Query Count PseudoCode

```
def RangeQueryCount(Q, T):
  if T == NULL: return 0
  if BB(T) doesn't overlap Query: return 0
  if Query subset of BB(T): return T.size

  count = 0
  if T.data in Query: count++

  count += RangeQuery(Q, t.left)
  count += RangeQuery(Q, t.right)

  return count
```

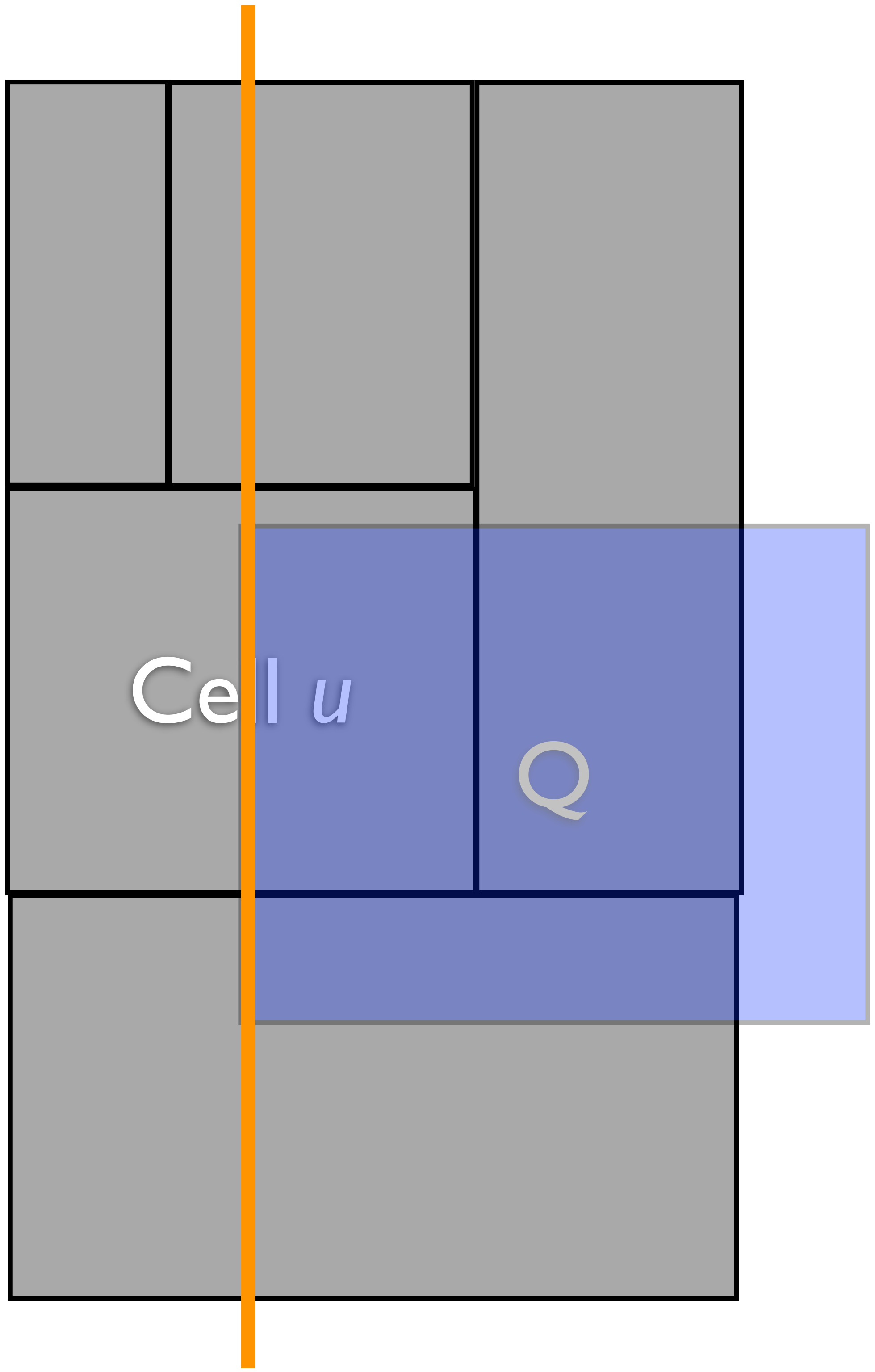
(For clarity, omitting the cutting dimension, and the BB(T) parameters that would be passed into the function)

Range Query PseudoCode

```
def RangeQuery(Q, T):
  if T == NULL: return empty_set()
  if BB(T) doesn't overlap Query: return 0
  if Query subset of BB(T): return AllNodesUnder(T)

  set = empty_set()
  if T.data in Query: set.union({T.data})

  set.union(RangeQuery(Q, T.left))
  set.union(RangeQuery(Q, T.right))

  return set
```

Expected # of Nodes to Visit

- Completely process a node only if query box intersects bounding box of the node's cell:
- In other words, one of the edges of Q must cut through the cell.
- # of cells a vertical line will pass through ≥ the number of cells cut by the left edge of Q.
- Top, bottom, right edges are the same, so bounding # of cells cut by a vertical line is sufficient.



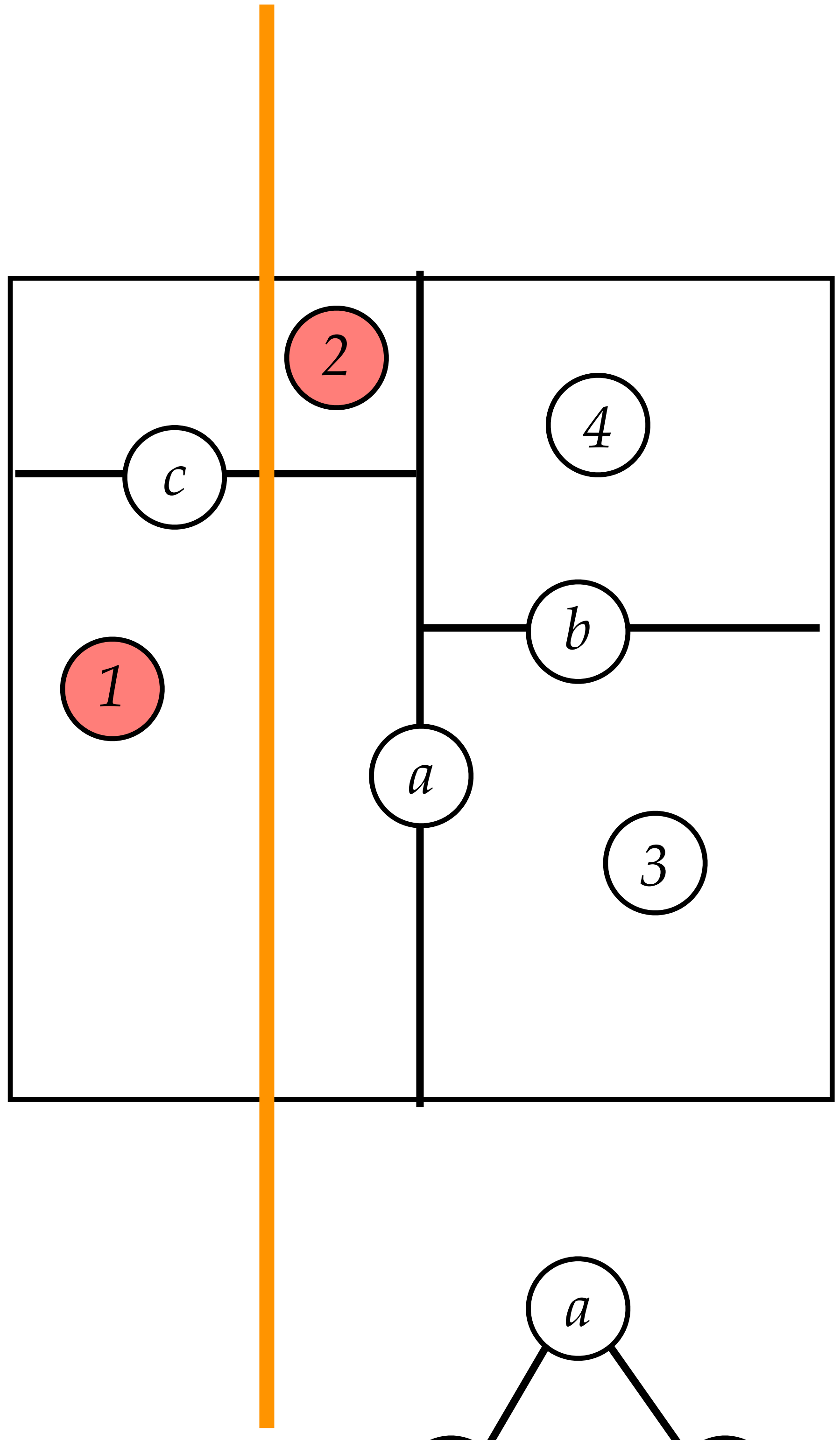
of Stabbed Nodes = O(√n)

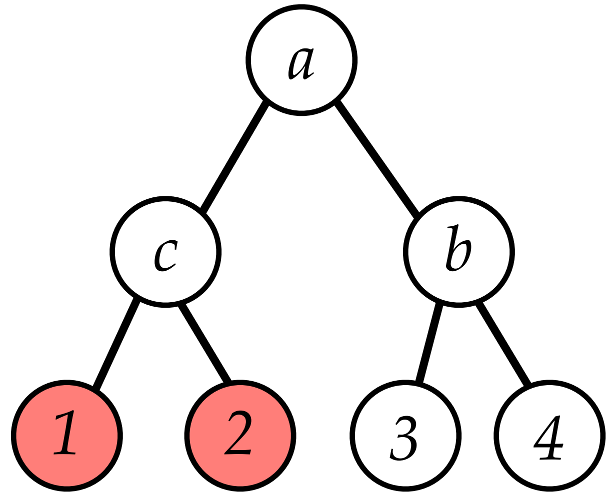
Consider a node *a* with cutting dimension = x

Vertical line can intersect exactly one of *a*'s children (say *c*)

But will intersect *both* of c's children.

Thus, line will intersect at most 2 of *a*'s grandchildren.





of Stabbed Nodes = O(√n)

So: you at most double # of cut nodes every 2 levels

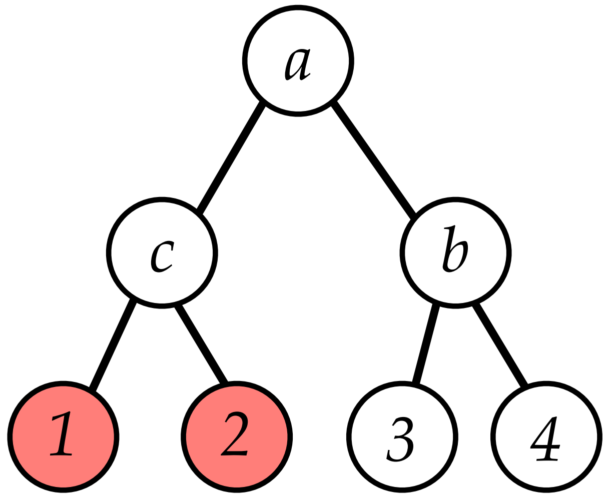
If kd-tree is balanced, has O(log n) levels

Cells cut

$= 2^{(\log n)/2}$

$= 2^{\log \sqrt{n}}$

$= \sqrt{n}$



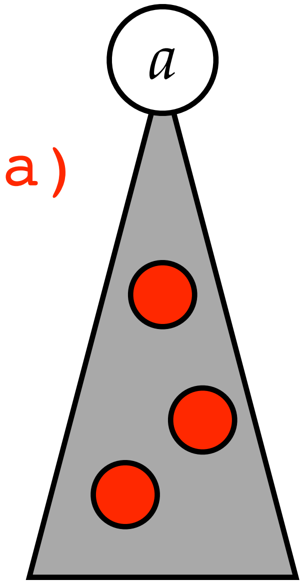
Assuming random input, or all points known ahead of time, you'll get a balanced tree.

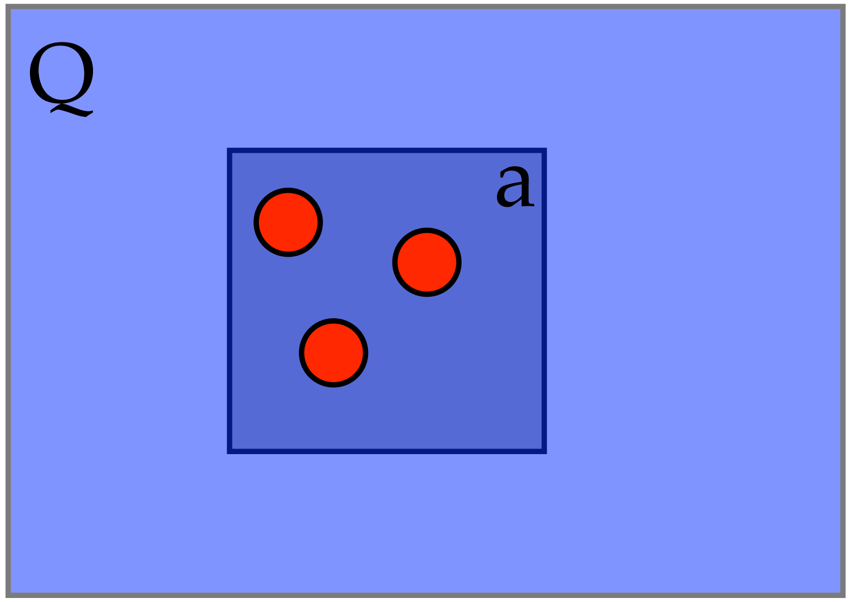
Each side of query rectangle stabs < O(√n) cells. So whole query stabs at most O(4√n) = O(√n) cells.

Suppose we want to output all points in region

- Then cost is O(k + √n)
  - where k is # of points in the query region.
- Why? Because: you visit every stabbed node [O(√n) of them] + every node in the subtrees rooted in the contained cells.
  - Takes linear time to traverse such subtrees
- Example of *<u>output sensitive</u>* running time analysis: running time depends on size of the *output*.

`AllNodesUnder(a)`





kd-tree Summary:

- Use $O(n)$ storage [1 node for each point]

- If all points are known in advance, balanced kd-tree can be built in $O(n \log n)$ time
  - Recall: sort the points by x and y coordinates
  - Always split on the median point so each split divides remaining points nearly in half.
  - Time dominated by the initial sorting.

- Can be orthogonal range searched in $O(\sqrt{n} + k)$ time.

- Can we do better than $O(\sqrt{n})$ to range search?
  - (possibly at a cost of additional space)