CMU SCS

15-826: Multimedia Databases and Data Mining

Lecture #18: SVD - part I (definitions)

C. Faloutsos

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Must-read Material

- Numerical Recipes in C ch. 2.6;
- Textbook Appendix D

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Outline

Goal: 'Find similar / interesting things'

- Intro to DB
- Indexing - similarity search
- Data Mining

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Indexing - Detailed outline

- primary key indexing
- secondary key / multi-key indexing
- spatial access methods
- fractals
- text
- Singular Value Decomposition (SVD)
- multimedia
- ...

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SVD - Detailed outline

- Motivation
- Definition - properties
- Interpretation
- Complexity
- Case studies
- Additional properties

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SVD - Motivation

- problem #1: text - LSI: find ‘concepts’
- problem #2: compression / dim. reduction

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SVD - Motivation

- problem #1: text - LSI: find 'concepts'

document \ term	data	information	retrieval	brain	lung
CS-TR1	1	1	1	0	0
CS-TR2	2	2	2	0	0
CS-TR3	1	1	1	0	0
CS-TR4	5	5	5	0	0
MED-TR1	0	0	0	2	2
MED-TR2	0	0	0	3	3
MED-TR3	0	0	0	1	1

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SVD - Motivation

- problem #2: compress / reduce dimensionality

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Problem - specs

- ~10**6 rows; ~10**3 columns; no updates;
- random access to any cell(s) ; small error: OK

customer \ day	We 7/10/96	Th 7/11/96	Fr 7/12/96	Sa 7/13/96	Su 7/14/96
ABC Inc.	1	1	1	0	0
DEF Ltd.	2	2	2	0	0
GHI Inc.	1	1	$\boxed{1}$	0	0
KLM Co.	5	5	5	0	0
Smith	0	0	0	2	2
Johnson	0	0	0	3	3
Thompson	0	0	0	1	1

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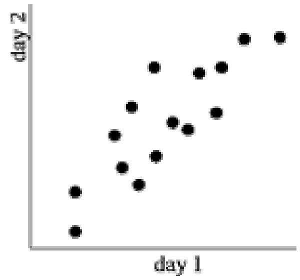
SVD - Motivation



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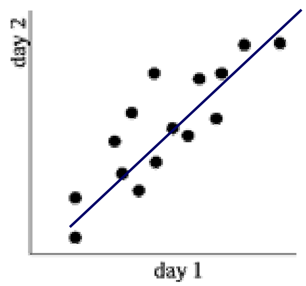
SVD - Motivation



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SVD - Detailed outline

- Motivation
- Definition - properties
- Interpretation
- Complexity
- Case studies
- Additional properties

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SVD - Definition

(reminder: matrix multiplication

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \text{x} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \ \\ \ \\ \ \end{bmatrix}$$

3 x *2* *2* x *1*

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SVD - Definition

(reminder: matrix multiplication

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \text{x} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \ \\ \ \\ \ \end{bmatrix}$$

3 x *2* *2* x *1* *3* x *1*

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SVD - Definition

(reminder: matrix multiplication

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \text{x} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ \ \\ \ \end{bmatrix}$$

3 x *2* *2* x *1* *3* x *1*

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SVD - Definition

(reminder: matrix multiplication

$$\underset{3 \times 2}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}} \times \underset{2 \times 1}{\begin{bmatrix} 1 \\ -1 \end{bmatrix}} = \underset{3 \times 1}{\begin{bmatrix} -1 \\ -1 \\ \end{bmatrix}}$$

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SVD - Definition

(reminder: matrix multiplication

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \times \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$$

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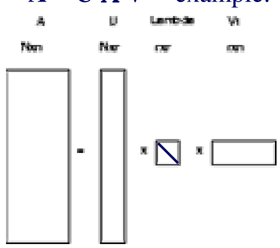
SVD - Definition

$$\boxed{\mathbf{A}_{[\mathbf{n} \times \mathbf{m}]} = \mathbf{U}_{[\mathbf{n} \times \mathbf{r}]} \, \mathbf{\Lambda}_{[\mathbf{r} \times \mathbf{r}]} \, (\mathbf{V}_{[\mathbf{m} \times \mathbf{r}]})^T}$$

- **A**: n x m matrix (eg., n documents, m terms)
- **U**: n x r matrix (n documents, r concepts)
- **Λ**: r x r diagonal matrix (strength of each 'concept') (r : rank of the matrix)
- **V**: m x r matrix (m terms, r concepts)

SVD - Definition

- $\mathbf{A} = \mathbf{U}\,\mathbf{\Lambda}\,\mathbf{V}^T$ - example:



SVD - Properties

THEOREM [Press+92]: always possible to decompose matrix $\mathbf{A}$ into $\mathbf{A} = \mathbf{U}\,\mathbf{\Lambda}\,\mathbf{V}^T$, where

- $\mathbf{U}, \mathbf{\Lambda}, \mathbf{V}$: unique (*)
- $\mathbf{U}$, $\mathbf{V}$: column orthonormal (ie., columns are unit vectors, orthogonal to each other)
  - $\mathbf{U}^T\,\mathbf{U} = \mathbf{I}$; $\mathbf{V}^T\,\mathbf{V} = \mathbf{I}$ (**I:** identity matrix)
- $\mathbf{\Lambda}$**:** singular are positive, and sorted in decreasing order

SVD - Example

- $\mathbf{A} = \mathbf{U}\,\mathbf{\Lambda}\,\mathbf{V}^T$ - example:

$$
\begin{array}{c c}
 & \begin{array}{ccccc} \text{data} & \text{inf.} & \text{retrieval} & \text{brain} & \text{lung} \end{array} \\
\begin{array}{c} \text{CS} \\ \\ \\ \\ \text{MD} \\ \\ \\ \end{array} &
\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}
\end{array}
=
\begin{bmatrix} 0.18 & 0 \\ 0.36 & 0 \\ 0.18 & 0 \\ 0.90 & 0 \\ 0 & 0.53 \\ 0 & 0.80 \\ 0 & 0.27 \end{bmatrix}
\times
\begin{bmatrix} 9.64 & 0 \\ 0 & 5.29 \end{bmatrix}
\times
\begin{bmatrix} 0.58 & 0.58 & 0.58 & 0 & 0 \\ 0 & 0 & 0 & 0.71 & 0.71 \end{bmatrix}
$$

SVD - Example

- **A = U Λ V**^{T} - example:

CS-concept

MD-concept

$$
\begin{array}{c|c}
 & \text{data} \quad \text{inf.} \quad \text{retrieval} \quad \text{brain} \quad \text{lung} \\
\begin{array}{c} \text{CS} \\ \\ \\ \\ \text{MD} \\ \\ \\ \end{array} &
\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}
= \begin{bmatrix} 0.18 & 0 \\ 0.36 & 0 \\ 0.18 & 0 \\ 0.90 & 0 \\ 0 & 0.53 \\ 0 & 0.80 \\ 0 & 0.27 \end{bmatrix}
\times \begin{bmatrix} 9.64 & 0 \\ 0 & 5.29 \end{bmatrix}
\times \begin{bmatrix} 0.58 & 0.58 & 0.58 & 0 & 0 \\ 0 & 0 & 0 & 0.71 & 0.71 \end{bmatrix}
\end{array}
$$

SVD - Example

- **A = U Λ V**^{T} - example:

doc-to-concept similarity matrix

CS-concept

MD-concept

$$
\begin{array}{c|c}
 & \text{data} \quad \text{inf.} \quad \text{retrieval} \quad \text{brain} \quad \text{lung} \\
\begin{array}{c} \text{CS} \\ \\ \\ \\ \text{MD} \\ \\ \\ \end{array} &
\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}
= \begin{bmatrix} 0.18 & 0 \\ 0.36 & 0 \\ 0.18 & 0 \\ 0.90 & 0 \\ 0 & 0.53 \\ 0 & 0.80 \\ 0 & 0.27 \end{bmatrix}
\times \begin{bmatrix} 9.64 & 0 \\ 0 & 5.29 \end{bmatrix}
\times \begin{bmatrix} 0.58 & 0.58 & 0.58 & 0 & 0 \\ 0 & 0 & 0 & 0.71 & 0.71 \end{bmatrix}
\end{array}
$$

SVD - Example

- **A = U Λ V**^{T} - example:

'strength' of CS-concept

$$
\begin{array}{c|c}
 & \text{data} \quad \text{inf.} \quad \text{retrieval} \quad \text{brain} \quad \text{lung} \\
\begin{array}{c} \text{CS} \\ \\ \\ \\ \text{MD} \\ \\ \\ \end{array} &
\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}
= \begin{bmatrix} 0.18 & 0 \\ 0.36 & 0 \\ 0.18 & 0 \\ 0.90 & 0 \\ 0 & 0.53 \\ 0 & 0.80 \\ 0 & 0.27 \end{bmatrix}
\times \begin{bmatrix} 9.64 & 0 \\ 0 & 5.29 \end{bmatrix}
\times \begin{bmatrix} 0.58 & 0.58 & 0.58 & 0 & 0 \\ 0 & 0 & 0 & 0.71 & 0.71 \end{bmatrix}
\end{array}
$$

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SVD - Example

- $\mathbf{A} = \mathbf{U}\, \mathbf{\Lambda}\, \mathbf{V}^T$ - example:

term-to-concept similarity matrix

CS-concept

$$
\begin{array}{c} \\ \text{CS} \\ \\ \text{MD} \end{array}
\begin{array}{c}
\begin{array}{ccccc} \text{data} & \text{inf.} & \text{retrieval} & \text{brain} & \text{lung} \end{array} \\
\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}
\end{array}
=
\begin{bmatrix} 0.18 & 0 \\ 0.36 & 0 \\ 0.18 & 0 \\ 0.90 & 0 \\ 0 & 0.53 \\ 0 & 0.80 \\ 0 & 0.27 \end{bmatrix}
\mathrm{x}
\begin{bmatrix} 9.64 & 0 \\ 0 & 5.29 \end{bmatrix}
\mathrm{x}
\begin{bmatrix} 0.58 & 0.58 & 0.58 & 0 & 0 \\ 0 & 0 & 0 & 0.71 & 0.71 \end{bmatrix}
$$

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SVD - Example

- $\mathbf{A} = \mathbf{U}\, \mathbf{\Lambda}\, \mathbf{V}^T$ - example:

term-to-concept similarity matrix

CS-concept

$$
\begin{array}{c} \\ \text{CS} \\ \\ \text{MD} \end{array}
\begin{array}{c}
\begin{array}{ccccc} \text{data} & \text{inf.} & \text{retrieval} & \text{brain} & \text{lung} \end{array} \\
\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}
\end{array}
=
\begin{bmatrix} 0.18 & 0 \\ 0.36 & 0 \\ 0.18 & 0 \\ 0.90 & 0 \\ 0 & 0.53 \\ 0 & 0.80 \\ 0 & 0.27 \end{bmatrix}
\mathrm{x}
\begin{bmatrix} 9.64 & 0 \\ 0 & 5.29 \end{bmatrix}
\mathrm{x}
\begin{bmatrix} 0.58 & 0.58 & 0.58 & 0 & 0 \\ 0 & 0 & 0 & 0.71 & 0.71 \end{bmatrix}
$$

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SVD - Detailed outline

- Motivation
- Definition - properties
- Interpretation
- Complexity
- Case studies
- Additional properties

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SVD - Interpretation #1

'documents', 'terms' and 'concepts':

- **U**: document-to-concept similarity matrix
- **V**: term-to-concept sim. matrix
- **Λ**: its diagonal elements: 'strength' of each concept

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SVD - Interpretation #2

- best axis to project on: ('best' = min sum of squares of projection errors)

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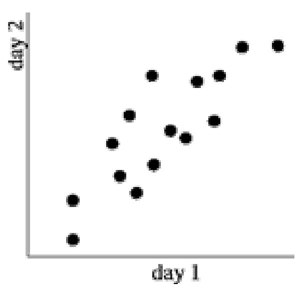
SVD - Motivation



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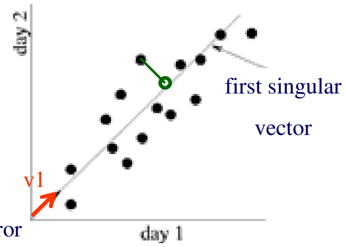
SVD - interpretation #2

SVD: gives best axis to project



- minimum RMS error

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SVD - Interpretation #2

day / customer	We 7/10/96	Th 7/11/96	Fr 7/12/96	Sa 7/13/96	Su 7/14/96
ABC Inc.	1	1	1	0	0
DEF Ltd.	2	2	2	0	0
GHI Inc.	1	1	1	0	0
KLM Co.	5	5	5	0	0
Smith	0	0	0	2	2
Johnson	0	0	0	3	3
Thompson	0	0	0	1	1

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SVD - Interpretation #2

- $\mathbf{A} = \mathbf{U}\, \mathbf{\Lambda}\, \mathbf{V}^T$ - example:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0.18 & 0 \\ 0.36 & 0 \\ 0.18 & 0 \\ 0.90 & 0 \\ 0 & 0.53 \\ 0 & 0.80 \\ 0 & 0.27 \end{bmatrix} \times \begin{bmatrix} 9.64 & 0 \\ 0 & 5.29 \end{bmatrix} \times \overset{v1}{\begin{bmatrix} \boxed{0.58 \;\; 0.58 \;\; 0.58 \;\; 0 \;\; 0} \\ 0 \;\; 0 \;\; 0 \;\; 0.71 \;\; 0.71 \end{bmatrix}}$$

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SVD - Interpretation #2

- **A = U Λ V**^{T} - example:

variance ('spread') on the v1 axis

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0.18 & 0 \\ 0.36 & 0 \\ 0.18 & 0 \\ 0.90 & 0 \\ 0 & 0.53 \\ 0 & 0.80 \\ 0 & 0.27 \end{bmatrix} \mathrm{x} \begin{bmatrix} 9.64 & 0 \\ 0 & 5.29 \end{bmatrix} \mathrm{x} \begin{bmatrix} 0.58 & 0.58 & 0.58 & 0 & 0 \\ 0 & 0 & 0 & 0.71 & 0.71 \end{bmatrix}$$

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SVD - Interpretation #2

- **A = U Λ V**^{T} - example:
  - **U Λ** gives the coordinates of the points in the projection axis

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0.18 & 0 \\ 0.36 & 0 \\ 0.18 & 0 \\ 0.90 & 0 \\ 0 & 0.53 \\ 0 & 0.80 \\ 0 & 0.27 \end{bmatrix} \mathrm{x} \begin{bmatrix} 9.64 & 0 \\ 0 & 5.29 \end{bmatrix} \mathrm{x} \begin{bmatrix} 0.58 & 0.58 & 0.58 & 0 & 0 \\ 0 & 0 & 0 & 0.71 & 0.71 \end{bmatrix}$$

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SVD - Interpretation #2

- More details
- Q: how exactly is dim. reduction done?

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0.18 & 0 \\ 0.36 & 0 \\ 0.18 & 0 \\ 0.90 & 0 \\ 0 & 0.53 \\ 0 & 0.80 \\ 0 & 0.27 \end{bmatrix} \mathrm{x} \begin{bmatrix} 9.64 & 0 \\ 0 & 5.29 \end{bmatrix} \mathrm{x} \begin{bmatrix} 0.58 & 0.58 & 0.58 & 0 & 0 \\ 0 & 0 & 0 & 0.71 & 0.71 \end{bmatrix}$$

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SVD - Interpretation #2

- More details
- Q: how exactly is dim. reduction done?
- A: set the smallest singular values to zero:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0.18 & 0 \\ 0.36 & 0 \\ 0.18 & 0 \\ 0.90 & 0 \\ 0 & 0.53 \\ 0 & 0.80 \\ 0 & 0.27 \end{bmatrix} \times \begin{bmatrix} 9.64 & 0 \\ 0 & \cancel{5.29} \end{bmatrix} \times \begin{bmatrix} 0.58 & 0.58 & 0.58 & 0 & 0 \\ 0 & 0 & 0 & 0.71 & 0.71 \end{bmatrix}$$

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SVD - Interpretation #2

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 0.18 & 0 \\ 0.36 & 0 \\ 0.18 & 0 \\ 0.90 & 0 \\ 0 & 0.53 \\ 0 & 0.80 \\ 0 & 0.27 \end{bmatrix} \times \begin{bmatrix} 9.64 & 0 \\ 0 & \cancel{0} \end{bmatrix} \times \begin{bmatrix} 0.58 & 0.58 & 0.58 & 0 & 0 \\ 0 & 0 & 0 & 0.71 & 0.71 \end{bmatrix}$$

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SVD - Interpretation #2

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 0.18 & \cancel{0} \\ 0.36 & \cancel{0} \\ 0.18 & \cancel{0} \\ 0.90 & \cancel{0} \\ 0 & \cancel{0.53} \\ 0 & \cancel{0.80} \\ 0 & \cancel{0.27} \end{bmatrix} \times \begin{bmatrix} 9.64 & 0 \\ 0 & \cancel{0} \end{bmatrix} \times \begin{bmatrix} 0.58 & 0.58 & 0.58 & 0 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0.71} & \cancel{0.71} \end{bmatrix}$$

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SVD - Interpretation #2

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 0.18 \\ 0.36 \\ 0.18 \\ 0.90 \\ 0 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 9.64 \end{bmatrix} \times \begin{bmatrix} 0.58 & 0.58 & 0.58 & 0 & 0 \end{bmatrix}$$

SVD - Interpretation #2

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

SVD - Interpretation #2

Exactly equivalent:
'spectral decomposition' of the matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0.18 & 0 \\ 0.36 & 0 \\ 0.18 & 0 \\ 0.90 & 0 \\ 0 & 0.53 \\ 0 & 0.80 \\ 0 & 0.27 \end{bmatrix} \times \begin{bmatrix} 9.64 & 0 \\ 0 & 5.29 \end{bmatrix} \times \begin{bmatrix} 0.58 & 0.58 & 0.58 & 0 & 0 \\ 0 & 0 & 0 & 0.71 & 0.71 \end{bmatrix}$$

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SVD - Interpretation #2

Exactly equivalent:
'spectral decomposition' of the matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \times \begin{bmatrix} \lambda_1 & \varnothing \\ \varnothing & \lambda_2 \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

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SVD - Interpretation #2

Exactly equivalent:
'spectral decomposition' of the matrix:

$$\begin{array}{cc} & \overleftrightarrow{\quad m \quad} \\ n & \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array} = \lambda_1 \; u_1 \; v^T_1 + \lambda_2 \; u_2 \; v^T_2 + \ldots$$

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SVD - Interpretation #2

Exactly equivalent:
'spectral decomposition' of the matrix:

$$\begin{array}{cc} & \overleftrightarrow{\quad m \quad} \\ n & \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array} = \overbrace{\lambda_1 \; \underbrace{u_1}_{n \times 1} \; \underbrace{v^T_1}_{1 \times m} + \lambda_2 \; u_2 \; v^T_2 + \ldots}^{r \text{ terms}}$$

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SVD - Interpretation #2

approximation / dim. reduction:
by keeping the first few terms (Q: how many?)

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}}_{m} n = \lambda_1 \; u_1 \; v^T_1 + \lambda_2 \; u_2 \; v^T_2 + \dots$$

assume: $\lambda_1 >= \lambda_2 >= \dots$

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SVD - Interpretation #2

A (heuristic - [Fukunaga]): keep 80-90% of 'energy' (= sum of squares of λ_i 's)

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}}_{m} n = \lambda_1 \; u_1 \; v^T_1 + \lambda_2 \; u_2 \; v^T_2 + \dots$$

assume: $\lambda_1 >= \lambda_2 >= \dots$

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SVD - Detailed outline

- Motivation
- Definition - properties
- Interpretation
  - #1: documents/terms/concepts
  - #2: dim. reduction
  - #3: picking non-zero, rectangular 'blobs'
- Complexity
- Case studies
- Additional properties

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SVD - Interpretation #3

- finds non-zero 'blobs' in a data matrix

$$\begin{bmatrix} 1&1&1&0&0\\ 2&2&2&0&0\\ 1&1&1&0&0\\ 5&5&5&0&0\\ 0&0&0&2&2\\ 0&0&0&3&3\\ 0&0&0&1&1 \end{bmatrix} = \begin{bmatrix} 0.18&0\\ 0.36&0\\ 0.18&0\\ 0.90&0\\ 0&0.53\\ 0&0.80\\ 0&0.27 \end{bmatrix} \times \begin{bmatrix} 9.64&0\\ 0&5.29 \end{bmatrix} \times \begin{bmatrix} 0.58&0.58&0.58&0&0\\ 0&0&0&0.71&0.71 \end{bmatrix}$$

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SVD - Interpretation #3

- finds non-zero 'blobs' in a data matrix

$$\left[\begin{array}{ccc|cc} 1&1&1&0&0\\ 2&2&2&0&0\\ 1&1&1&0&0\\ 5&5&5&0&0\\ \hline 0&0&0&2&2\\ 0&0&0&3&3\\ 0&0&0&1&1 \end{array}\right] = \begin{bmatrix} 0.18&0\\ 0.36&0\\ 0.18&0\\ 0.90&0\\ 0&0.53\\ 0&0.80\\ 0&0.27 \end{bmatrix} \times \begin{bmatrix} 9.64&0\\ 0&5.29 \end{bmatrix} \times \begin{bmatrix} 0.58&0.58&0.58&0&0\\ 0&0&0&0.71&0.71 \end{bmatrix}$$

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SVD - Interpretation #3

- Drill: find the SVD, 'by inspection'!
- Q: rank = ??

$$\begin{bmatrix} 1&1&1&0&0\\ 1&1&1&0&0\\ 1&1&1&0&0\\ 0&0&0&1&1\\ 0&0&0&1&1 \end{bmatrix} = \begin{bmatrix} ?? \end{bmatrix} \times \begin{bmatrix} ?? \end{bmatrix} \times \begin{bmatrix} ?? \end{bmatrix}$$

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SVD - Interpretation #3

- A: rank = 2 (2 linearly independent rows/cols)

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} | & | \\ ?? & ?? \\ | & | \end{bmatrix} \times \begin{bmatrix} ?? & 0 \\ 0 & ?? \end{bmatrix} \times \begin{bmatrix} \text{——— ?? ———} \\ \text{——— ?? ———} \end{bmatrix}$$

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SVD - Interpretation #3

- A: rank = 2 (2 linearly independent rows/cols)

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} ?? & 0 \\ 0 & ?? \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

orthogonal??

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SVD - Interpretation #3

- column vectors: are orthogonal - but not unit vectors:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/sqrt(3) & 0 \\ 1/sqrt(3) & 0 \\ 1/sqrt(3) & 0 \\ 0 & 1/sqrt(2) \\ 0 & 1/sqrt(2) \end{bmatrix} \times \begin{bmatrix} ?? & 0 \\ 0 & ?? \end{bmatrix} \times \begin{bmatrix} 1/sqrt(3) & 1/sqrt(3) & 1/sqrt(3) & 0 & 0 \\ 0 & 0 & 0 & 1/sqrt(2) & 1/sqrt(2) \end{bmatrix}$$

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SVD - Interpretation #3

- and the singular values are:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/sqrt(3) & 0 \\ 1/sqrt(3) & 0 \\ 1/sqrt(3) & 0 \\ 0 & 1/sqrt(2) \\ 0 & 1/sqrt(2) \end{bmatrix} \times \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \times \begin{bmatrix} 1/sqrt(3) & 1/sqrt(3) & 1/sqrt(3) & 0 & 0 \\ 0 & 0 & 0 & 1/sqrt(2) & 1/sqrt(2) \end{bmatrix}$$

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SVD - Interpretation #3

- Q: How to check we are correct?

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/sqrt(3) & 0 \\ 1/sqrt(3) & 0 \\ 1/sqrt(3) & 0 \\ 0 & 1/sqrt(2) \\ 0 & 1/sqrt(2) \end{bmatrix} \times \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \times \begin{bmatrix} 1/sqrt(3) & 1/sqrt(3) & 1/sqrt(3) & 0 & 0 \\ 0 & 0 & 0 & 1/sqrt(2) & 1/sqrt(2) \end{bmatrix}$$

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SVD - Interpretation #3

- A: SVD properties:
  - matrix product should give back matrix **A**
  - matrix **U** should be column-orthonormal, i.e., columns should be unit vectors, orthogonal to each other
  - ditto for matrix **V**
  - matrix **Λ** should be diagonal, with positive values

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SVD - Detailed outline

- Motivation
- Definition - properties
- Interpretation
- Complexity
- Case studies
- Additional properties

SVD - Complexity

- O(n * m * m) or O(n * n * m) (whichever is less)
- less work, if we just want singular values
- or if we want first *k* singular vectors
- or if the matrix is sparse [Berry]
- Implemented: in any linear algebra package (LINPACK, matlab, Splus, mathematica ...)

SVD - conclusions so far

- SVD: $\mathbf{A} = \mathbf{U} \, \mathbf{\Lambda} \, \mathbf{V}^T$: unique (*)
- **U**: document-to-concept similarities
- **V**: term-to-concept similarities
- $\mathbf{\Lambda}$: strength of each concept
- dim. reduction: keep the first few strongest singular values (80-90% of 'energy')
  - SVD: picks up linear correlations
- SVD: picks up non-zero 'blobs'

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