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15-826: Multimedia Databases and Data Mining

Lecture #16: SVD - part I (definitions)

C. Faloutsos

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Must-read Material

- [Numerical Recipes in C](#) ch. 2.6;
- [MM Textbook](#) Appendix D

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Outline


Goal: 'Find **similar / interesting things**'

- Intro to DB
-  • Indexing - similarity search
-  • Data Mining

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Indexing - Detailed outline

- primary key indexing
- secondary key / multi-key indexing
- spatial access methods
- fractals
- text
-  • Singular Value Decomposition (SVD)
- multimedia
- ...

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SVD - Detailed outline



- ➔ • Motivation
- Definition - properties
- Interpretation
- Complexity
- Case studies
- Additional properties

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Problem

- How to find ‘concepts’ in a set of doc’s?
 - (~ clusters)



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Conclusion

- How to find ‘concepts’ in a set of doc’s?
 - (~ clusters)
- SVD (= LSI) on the document-term matrix

A	U	Lambda	V ^t
N _{doc} × N _{term}	N _{doc} × r	r × r	r × N _{term}
=		×	×



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SVD - Motivation

- problem #1: text - LSI: find ‘concepts’
- problem #2: compression / dim. reduction

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SVD - Motivation

- problem #1: text - LSI: find 'concepts'

document	term	data	information	retrieval	brain	lung
CS-TR1		1	1	1	0	0
CS-TR2		2	2	2	0	0
CS-TR3		1	1	1	0	0
CS-TR4		5	5	5	0	0
MED-TR1		0	0	0	2	2
MED-TR2		0	0	0	3	3
MED-TR3		0	0	0	1	1

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SVD - Motivation

- Customer-product, for recommendation system:

		bread	lettuce	tomatos	beef	chicken
↑	↑	$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$				
↓	↓					
↑	↑					
↓	↓					
↑	↑					
↓	↓					

↑ vegetarians
↑ meat eaters

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SVD - Motivation

- problem #2: compress / reduce dimensionality

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Problem - specs

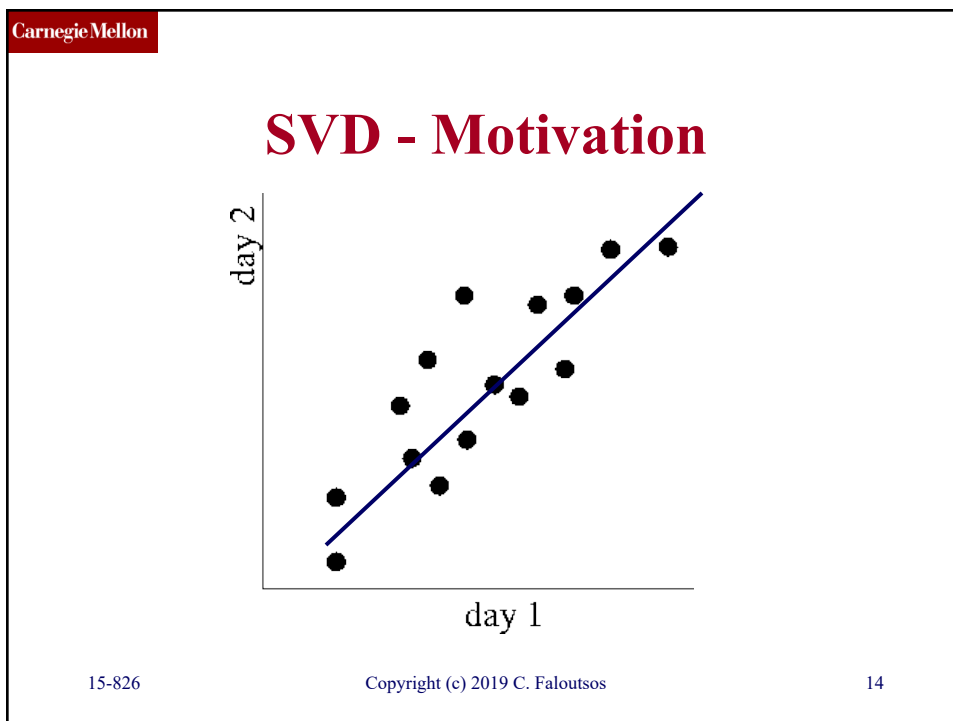
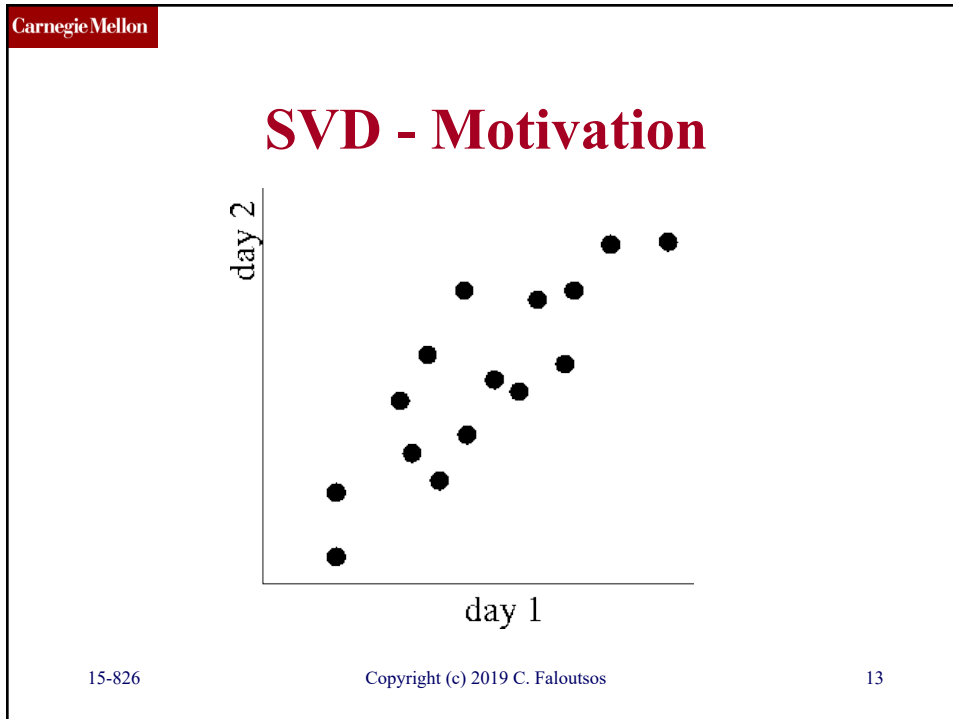
- $\sim 10 \times 6$ rows; $\sim 10 \times 3$ columns; no updates;
- random access to any cell(s) ; small error: OK

customer	day	We	Th	Fr	Sa	Su
		7/10/96	7/11/96	7/12/96	7/13/96	7/14/96
ABC Inc.		1	1	1	0	0
DEF Ltd.		2	2	2	0	0
GHI Inc.		1	1	1	0	0
KLM Co.		5	5	5	0	0
Smith		0	0	0	2	2
Johnson		0	0	0	3	3
Thompson		0	0	0	1	1

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SVD - Detailed outline

- Motivation
- ➔ • Definition - properties
- Interpretation
- Complexity
- Case studies
- Additional properties

SVD - Definition

(reminder: matrix multiplication)

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \times \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \\ \\ \end{bmatrix}$$

3×2 2×1

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SVD - Definition

(reminder: matrix multiplication)

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \times \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \\ \\ \end{bmatrix}$$

$\xleftarrow{\quad} \begin{matrix} 3 \times 2 & & 2 \times 1 & & 3 \times 1 \end{matrix} \xrightarrow{\quad}$

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SVD - Definition

(reminder: matrix multiplication)

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \times \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ \\ \end{bmatrix}$$

$\xleftarrow{\quad} \begin{matrix} 3 \times 2 & & 2 \times 1 & & 3 \times 1 \end{matrix} \xrightarrow{\quad}$

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SVD - Definition

(reminder: matrix multiplication)

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \times \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$$

$\xleftarrow{\quad} \begin{matrix} 3 \times 2 & & 2 \times 1 & & 3 \times 1 \\ & \xrightarrow{\quad} & & \xrightarrow{\quad} & \end{matrix}$

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SVD - Definition

(reminder: matrix multiplication)

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \times \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$$

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SVD - Definition

- $A = U \Lambda V^T$ - example:

A	U	Λ	V^T
$n \times m$	$n \times r$	$r \times r$	$m \times r$
			

- x x

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SVD - Definition

$$A_{[n \times m]} = U_{[n \times r]} \Lambda_{[r \times r]} (V_{[m \times r]})^T$$

- A : $n \times m$ matrix (eg., n documents, m terms)
- U : $n \times r$ matrix (n documents, r concepts)
- Λ : $r \times r$ diagonal matrix (strength of each 'concept') (r : rank of the matrix)
- V : $m \times r$ matrix (m terms, r concepts)

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SVD - Properties

THEOREM [Press+92]: always possible to decompose matrix \mathbf{A} into $\mathbf{A} = \mathbf{U} \mathbf{\Lambda} \mathbf{V}^T$, where

- $\mathbf{U}, \mathbf{\Lambda}, \mathbf{V}$: unique (*)
- \mathbf{U}, \mathbf{V} : column orthonormal (ie., columns are unit vectors, orthogonal to each other)
 - $\mathbf{U}^T \mathbf{U} = \mathbf{I}; \mathbf{V}^T \mathbf{V} = \mathbf{I}$ (\mathbf{I} : identity matrix)
- $\mathbf{\Lambda}$: singular are positive, and sorted in decreasing order

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SVD - Example

- $\mathbf{A} = \mathbf{U} \mathbf{\Lambda} \mathbf{V}^T$ - example:

$$\begin{array}{c}
 \begin{array}{c} \uparrow \\ \text{CS} \\ \downarrow \end{array} \\
 \begin{array}{c} \uparrow \\ \text{MD} \\ \downarrow \end{array}
 \end{array}
 \begin{array}{c}
 \text{retrieval} \\
 \text{data} \quad \text{inf.} \downarrow \quad \text{brain} \quad \text{lung} \\
 \begin{bmatrix}
 1 & 1 & 1 & 0 & 0 \\
 2 & 2 & 2 & 0 & 0 \\
 1 & 1 & 1 & 0 & 0 \\
 5 & 5 & 5 & 0 & 0 \\
 0 & 0 & 0 & 2 & 2 \\
 0 & 0 & 0 & 3 & 3 \\
 0 & 0 & 0 & 1 & 1
 \end{bmatrix}
 =
 \begin{bmatrix}
 0.18 & 0 \\
 0.36 & 0 \\
 0.18 & 0 \\
 0.90 & 0 \\
 0 & 0.53 \\
 0 & 0.80 \\
 0 & 0.27
 \end{bmatrix}
 \times
 \begin{bmatrix}
 9.64 & 0 \\
 0 & 5.29
 \end{bmatrix}
 \times
 \begin{bmatrix}
 0.58 & 0.58 & 0.58 & 0 & 0 \\
 0 & 0 & 0 & 0.71 & 0.71
 \end{bmatrix}
 \end{array}$$

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SVD - Example

• $A = U \Lambda V^T$ - example:

			retrieval	CS-concept			
	data	inf. ↓	brain	lung	MD-concept		
↑ CS	$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$	=	$\begin{bmatrix} 0.18 & 0 \\ 0.36 & 0 \\ 0.18 & 0 \\ 0.90 & 0 \\ 0 & 0.53 \\ 0 & 0.80 \\ 0 & 0.27 \end{bmatrix}$	x	$\begin{bmatrix} 9.64 & 0 \\ 0 & 5.29 \end{bmatrix}$	x	$\begin{bmatrix} 0.58 & 0.58 & 0.58 & 0 & 0 \\ 0 & 0 & 0 & 0.71 & 0.71 \end{bmatrix}$
↓ MD							

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SVD - Example

• $A = U \Lambda V^T$ - example: doc-to-concept similarity matrix

			retrieval	CS-concept			
	data	inf. ↓	brain	lung	MD-concept		
↑ CS	$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$	=	$\begin{bmatrix} 0.18 & 0 \\ 0.36 & 0 \\ 0.18 & 0 \\ 0.90 & 0 \\ 0 & 0.53 \\ 0 & 0.80 \\ 0 & 0.27 \end{bmatrix}$	x	$\begin{bmatrix} 9.64 & 0 \\ 0 & 5.29 \end{bmatrix}$	x	$\begin{bmatrix} 0.58 & 0.58 & 0.58 & 0 & 0 \\ 0 & 0 & 0 & 0.71 & 0.71 \end{bmatrix}$
↓ MD							

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SVD - Example

• $A = U \Lambda V^T$ - example:

retrieval
inf. ↓
data brain lung

↑ CS
↓
↑ MD
↓

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$=$

$$\begin{bmatrix} 0.18 & 0 \\ 0.36 & 0 \\ 0.18 & 0 \\ 0.90 & 0 \\ 0 & 0.53 \\ 0 & 0.80 \\ 0 & 0.27 \end{bmatrix}$$

\times

$$\begin{bmatrix} 9.64 & 0 \\ 0 & 5.29 \end{bmatrix}$$

\times

$$\begin{bmatrix} 0.58 & 0.58 & 0.58 & 0 & 0 \\ 0 & 0 & 0 & 0.71 & 0.71 \end{bmatrix}$$

‘strength’ of CS-concept

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SVD - Example

• $A = U \Lambda V^T$ - example:

retrieval
inf. ↓
data brain lung

↑ CS
↓
↑ MD
↓

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$=$

$$\begin{bmatrix} 0.18 & 0 \\ 0.36 & 0 \\ 0.18 & 0 \\ 0.90 & 0 \\ 0 & 0.53 \\ 0 & 0.80 \\ 0 & 0.27 \end{bmatrix}$$

\times

$$\begin{bmatrix} 9.64 & 0 \\ 0 & 5.29 \end{bmatrix}$$

\times

$$\begin{bmatrix} 0.58 & 0.58 & 0.58 & 0 & 0 \\ 0 & 0 & 0 & 0.71 & 0.71 \end{bmatrix}$$

term-to-concept
similarity matrix

CS-concept

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SVD - Interpretation #1

‘documents’, ‘terms’ and ‘concepts’ :

- **U**: document-to-concept similarity matrix
- **V**: term-to-concept sim. matrix
- **Λ** : its diagonal elements: ‘strength’ of each concept

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SVD – Interpretation #1

‘documents’, ‘terms’ and ‘concepts’ :

Q: if **A** is the document-to-term matrix, what is $\mathbf{A}^T \mathbf{A}$?

A:

Q: $\mathbf{A} \mathbf{A}^T$?

A:

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SVD – Interpretation #1

‘documents’, ‘terms’ and ‘concepts’ :

Q: if \mathbf{A} is the document-to-term matrix, what is $\mathbf{A}^T \mathbf{A}$?

A: term-to-term ($[m \times m]$) similarity matrix

Q: $\mathbf{A} \mathbf{A}^T$?

A: document-to-document ($[n \times n]$) similarity matrix

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SVD properties

- \mathbf{V} are the eigenvectors of the *covariance matrix* $\mathbf{A}^T \mathbf{A}$
- \mathbf{U} are the eigenvectors of the *Gram (inner-product) matrix* $\mathbf{A} \mathbf{A}^T$

Further reading:

1. Ian T. Jolliffe, *Principal Component Analysis* (2nd ed), Springer, 2002.
2. Gilbert Strang, *Linear Algebra and Its Applications* (4th ed), Brooks Cole, 2005.

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SVD - Interpretation #2

- best axis to project on: (‘best’ = min sum of squares of projection errors)

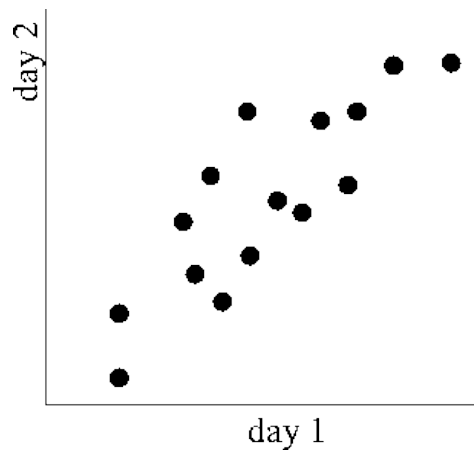
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SVD - Motivation



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SVD - interpretation #2

SVD: gives best axis to project

- minimum RMS error

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SVD - Interpretation #2

day	We	Th	Fr	Sa	Su
customer	7/10/96	7/11/96	7/12/96	7/13/96	7/14/96
ABC Inc.	1	1	1	0	0
DEF Ltd.	2	2	2	0	0
GHI Inc.	1	1	1	0	0
KLM Co.	5	5	5	0	0
Smith	0	0	0	2	2
Johnson	0	0	0	3	3
Thompson	0	0	0	1	1

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SVD - Interpretation #2

- $A = U \Lambda V^T$ - example:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0.18 & 0 \\ 0.36 & 0 \\ 0.18 & 0 \\ 0.90 & 0 \\ 0 & 0.53 \\ 0 & 0.80 \\ 0 & 0.27 \end{bmatrix} \times \begin{bmatrix} 9.64 & 0 \\ 0 & 5.29 \end{bmatrix} \times \begin{bmatrix} 0.58 & 0.58 & 0.58 & 0 & 0 \\ 0 & 0 & 0 & 0.71 & 0.71 \end{bmatrix}$$

v1

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SVD - Interpretation #2

- $A = U \Lambda V^T$ - example:

variance ('spread') on the $v1$ axis

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0.18 & 0 \\ 0.36 & 0 \\ 0.18 & 0 \\ 0.90 & 0 \\ 0 & 0.53 \\ 0 & 0.80 \\ 0 & 0.27 \end{bmatrix} \times \begin{bmatrix} 9.64 & 0 \\ 0 & 5.29 \end{bmatrix} \times \begin{bmatrix} 0.58 & 0.58 & 0.58 & 0 & 0 \\ 0 & 0 & 0 & 0.71 & 0.71 \end{bmatrix}$$

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SVD - Interpretation #2

- $A = U \Lambda V^T$ - example:
 - $U \Lambda$ gives the coordinates of the points in the projection axis

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0.18 & 0 \\ 0.36 & 0 \\ 0.18 & 0 \\ 0.90 & 0 \\ 0 & 0.53 \\ 0 & 0.80 \\ 0 & 0.27 \end{bmatrix} \times \begin{bmatrix} 9.64 & 0 \\ 0 & 5.29 \end{bmatrix} \times \begin{bmatrix} 0.58 & 0.58 & 0.58 & 0 & 0 \\ 0 & 0 & 0 & 0.71 & 0.71 \end{bmatrix}$$

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SVD - Interpretation #2

- More details
- Q: how exactly is dim. reduction done?

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0.18 & 0 \\ 0.36 & 0 \\ 0.18 & 0 \\ 0.90 & 0 \\ 0 & 0.53 \\ 0 & 0.80 \\ 0 & 0.27 \end{bmatrix} \times \begin{bmatrix} 9.64 & 0 \\ 0 & 5.29 \end{bmatrix} \times \begin{bmatrix} 0.58 & 0.58 & 0.58 & 0 & 0 \\ 0 & 0 & 0 & 0.71 & 0.71 \end{bmatrix}$$

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SVD - Interpretation #2

- More details
- Q: how exactly is dim. reduction done?
- A: set the smallest singular values to zero:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0.18 & 0 \\ 0.36 & 0 \\ 0.18 & 0 \\ 0.90 & 0 \\ 0 & 0.53 \\ 0 & 0.80 \\ 0 & 0.27 \end{bmatrix} \times \begin{bmatrix} 9.64 & 0 \\ 0 & \cancel{5.29} \end{bmatrix} \times \begin{bmatrix} 0.58 & 0.58 & 0.58 & 0 & 0 \\ 0 & 0 & 0 & 0.71 & 0.71 \end{bmatrix}$$

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SVD - Interpretation #2

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 0.18 & 0 \\ 0.36 & 0 \\ 0.18 & 0 \\ 0.90 & 0 \\ 0 & 0.53 \\ 0 & 0.80 \\ 0 & 0.27 \end{bmatrix} \times \begin{bmatrix} 9.64 & 0 \\ 0 & \cancel{0} \end{bmatrix} \times \begin{bmatrix} 0.58 & 0.58 & 0.58 & 0 & 0 \\ 0 & 0 & 0 & 0.71 & 0.71 \end{bmatrix}$$

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SVD - Interpretation #2

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 0.18 & 0 \\ 0.36 & 0 \\ 0.18 & 0 \\ 0.90 & 0 \\ 0 & 0.53 \\ 0 & 0.30 \\ 0 & 0.27 \end{bmatrix} \times \begin{bmatrix} 9.64 & 0 \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0.58 & 0.58 & 0.58 & 0 & 0 \\ 0 & 0 & 0 & 0.71 & 0.71 \end{bmatrix}$$

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SVD - Interpretation #2

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 0.18 \\ 0.36 \\ 0.18 \\ 0.90 \\ 0 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 9.64 \end{bmatrix} \times \begin{bmatrix} 0.58 & 0.58 & 0.58 & 0 & 0 \end{bmatrix}$$

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SVD - Interpretation #2

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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SVD - Interpretation #2

Exactly equivalent:
‘spectral decomposition’ of the matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0.18 & 0 \\ 0.36 & 0 \\ 0.18 & 0 \\ 0.90 & 0 \\ 0 & 0.53 \\ 0 & 0.80 \\ 0 & 0.27 \end{bmatrix} \times \begin{bmatrix} 9.64 & 0 \\ 0 & 5.29 \end{bmatrix} \times \begin{bmatrix} 0.58 & 0.58 & 0.58 & 0 & 0 \\ 0 & 0 & 0 & 0.71 & 0.71 \end{bmatrix}$$

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Preliminaries: Spectral form

$$\mathbf{U} \times \mathbf{V}^T = u_1 \times v_1^T + u_2 \times v_2^T$$

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SVD – Spectral form

Exactly equivalent:
 ‘spectral decomposition’ of the matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} | & | \\ \mathbf{u}_1 & \mathbf{u}_2 \\ | & | \end{bmatrix} \times \begin{bmatrix} \lambda_1 & \emptyset \\ \emptyset & \lambda_2 \end{bmatrix} \times \begin{bmatrix} \text{---} & \mathbf{v}_1 & \text{---} \\ \text{---} & \mathbf{v}_2 & \text{---} \end{bmatrix}$$

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SVD – Spectral form

Exactly equivalent:
‘spectral decomposition’ of the matrix:

$$\begin{array}{c} \uparrow \\ n \\ \downarrow \end{array} \begin{array}{c} \leftarrow m \rightarrow \\ \left[\begin{array}{cccccc} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right] \end{array} = \lambda_1 \begin{array}{c} \text{u}_1 \\ \text{v}_1^T \end{array} + \lambda_2 \begin{array}{c} \text{u}_2 \\ \text{v}_2^T \end{array} + \dots$$

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SVD – Spectral form

Exactly equivalent:
‘spectral decomposition’ of the matrix:

$$\begin{array}{c} \uparrow \\ n \\ \downarrow \end{array} \begin{array}{c} \leftarrow m \rightarrow \\ \left[\begin{array}{cccccc} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right] \end{array} = \begin{array}{c} \leftarrow r \text{ terms} \rightarrow \\ \lambda_1 \begin{array}{c} \text{u}_1 \\ \text{v}_1^T \end{array} + \lambda_2 \begin{array}{c} \text{u}_2 \\ \text{v}_2^T \end{array} + \dots \\ \begin{array}{c} \nearrow \\ n \times 1 \end{array} \quad \begin{array}{c} \nwarrow \\ 1 \times m \end{array} \end{array}$$

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SVD – Spectral form

approximation / dim. reduction:
by keeping the first few terms (Q: how many?)

$$\begin{array}{c} \uparrow \\ n \\ \downarrow \end{array} \begin{array}{c} \leftarrow m \rightarrow \\ \left[\begin{array}{cccccc} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right] \end{array} = \lambda_1 u_1 v_1^T + \lambda_2 u_2 v_2^T + \dots$$

assume: $\lambda_1 \geq \lambda_2 \geq \dots$

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SVD - Spectral form

A (heuristic - [Fukunaga]): keep 80-90% of
'energy' (= sum of squares of λ_i 's)

$$\begin{array}{c} \uparrow \\ n \\ \downarrow \end{array} \begin{array}{c} \leftarrow m \rightarrow \\ \left[\begin{array}{cccccc} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right] \end{array} = \lambda_1 u_1 v_1^T + \lambda_2 u_2 v_2^T + \dots$$

assume: $\lambda_1 \geq \lambda_2 \geq \dots$

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SVD - Spectral form

Also, if there are clear blocks, each corresponds to a 'concept'/singular-vector-pair

$$\begin{matrix} \xleftarrow{m} & \xrightarrow{m} \\ \uparrow n & \downarrow n \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{matrix}$$

=

$$\lambda_1 \underbrace{u_1 v_1^T} + \lambda_2 \underbrace{u_2 v_2^T} + \dots$$

assume: $\lambda_1 \geq \lambda_2 \geq \dots$

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SVD - Spectral form

$U \times V^T$

=

$u_1 \times v_1^T$

+

$u_2 \times v_2^T$

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SVD – Spectral form

Arithmetic example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \times \begin{bmatrix} 10 & 2 \\ -10 & -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \times [10 \ 2] +$$

$$\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \times [-10 \ -2]$$

$$= \begin{bmatrix} -10 & -2 \\ -10 & -2 \\ -10 & -2 \end{bmatrix}$$

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SVD - Detailed outline

- Motivation
- Definition - properties
- Interpretation
 - #1: documents/terms/concepts
 - #2: dim. reduction
 - – #3: picking non-zero, rectangular ‘blobs’
- Complexity
- Case studies
- Additional properties

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SVD - Interpretation #3

- finds non-zero 'blobs' in a data matrix

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0.18 & 0 \\ 0.36 & 0 \\ 0.18 & 0 \\ 0.90 & 0 \\ 0 & 0.53 \\ 0 & 0.80 \\ 0 & 0.27 \end{bmatrix} \times \begin{bmatrix} 9.64 & 0 \\ 0 & 5.29 \end{bmatrix} \times \begin{bmatrix} 0.58 & 0.58 & 0.58 & 0 & 0 \\ 0 & 0 & 0 & 0.71 & 0.71 \end{bmatrix}$$

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SVD - Interpretation #3

- finds non-zero 'blobs' in a data matrix

$$\begin{bmatrix} 1 & 1 & 1 & | & 0 & 0 \\ 2 & 2 & 2 & | & 0 & 0 \\ 1 & 1 & 1 & | & 0 & 0 \\ 5 & 5 & 5 & | & 0 & 0 \\ \hline 0 & 0 & 0 & | & 2 & 2 \\ 0 & 0 & 0 & | & 3 & 3 \\ 0 & 0 & 0 & | & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0.18 & 0 \\ 0.36 & 0 \\ 0.18 & 0 \\ 0.90 & 0 \\ 0 & 0.53 \\ 0 & 0.80 \\ 0 & 0.27 \end{bmatrix} \times \begin{bmatrix} 9.64 & 0 \\ 0 & 5.29 \end{bmatrix} \times \begin{bmatrix} 0.58 & 0.58 & 0.58 & 0 & 0 \\ 0 & 0 & 0 & 0.71 & 0.71 \end{bmatrix}$$

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SVD - Interpretation #3

- finds non-zero ‘blobs’ in a data matrix =
- ‘communities’ (bi-partite cores, here)

1	1	1	0	0	Row 1	○	◆	Col 1
2	2	2	0	0	Row 4	○	◆	Col 3
1	1	1	0	0	Row 5	○	◆	Col 4
5	5	5	0	0	Row 7	○	◆	
0	0	0	2	2				
0	0	0	3	3				
0	0	0	1	1				

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SVD – Interpretation #3

- Another pictorial example
- And its connection to EigenSpokes

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SVD – Interpretation #3

- It finds blocks

The diagram illustrates the SVD decomposition of a matrix M (with N fans and M idols). The matrix is decomposed into vectors \vec{u}_1 and \vec{v}_1 . The vector \vec{u}_1 is associated with 'music lovers' and 'singers', \vec{v}_1 with 'sports lovers' and 'athletes', and \vec{u}_i with 'citizens' and 'politicians'.

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SVD – Interpretation #3

- It finds blocks
- A) Even if shuffled!**

The diagram illustrates the SVD decomposition of a matrix M (with N fans and M idols) where the matrix is shuffled. The matrix is decomposed into vectors \vec{u}_1 and \vec{v}_1 . The vector \vec{u}_1 is associated with 'music lovers' and 'singers', \vec{v}_1 with 'sports lovers' and 'athletes', and \vec{u}_i with 'citizens' and 'politicians'.

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SVD – Interpretation #3

- It finds blocks

B) Even if ‘salt+pepper’ noise

‘music lovers’ ‘singers’ ‘sports lovers’ ‘athletes’ ‘citizens’ ‘politicians’
 \vec{u}_1 \vec{v}_1 \vec{u}_i

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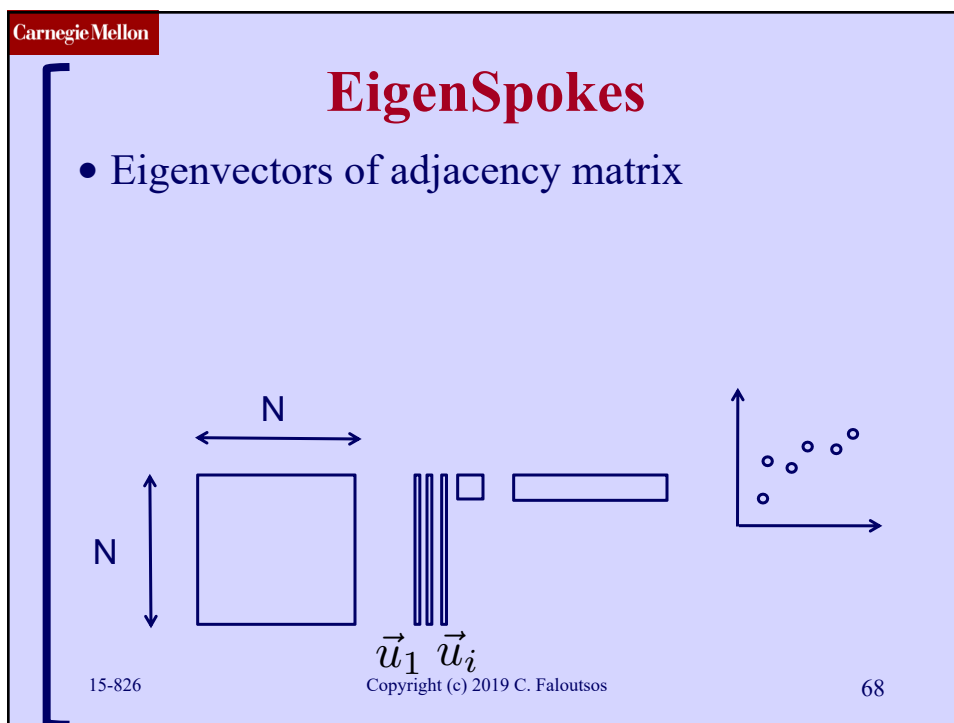
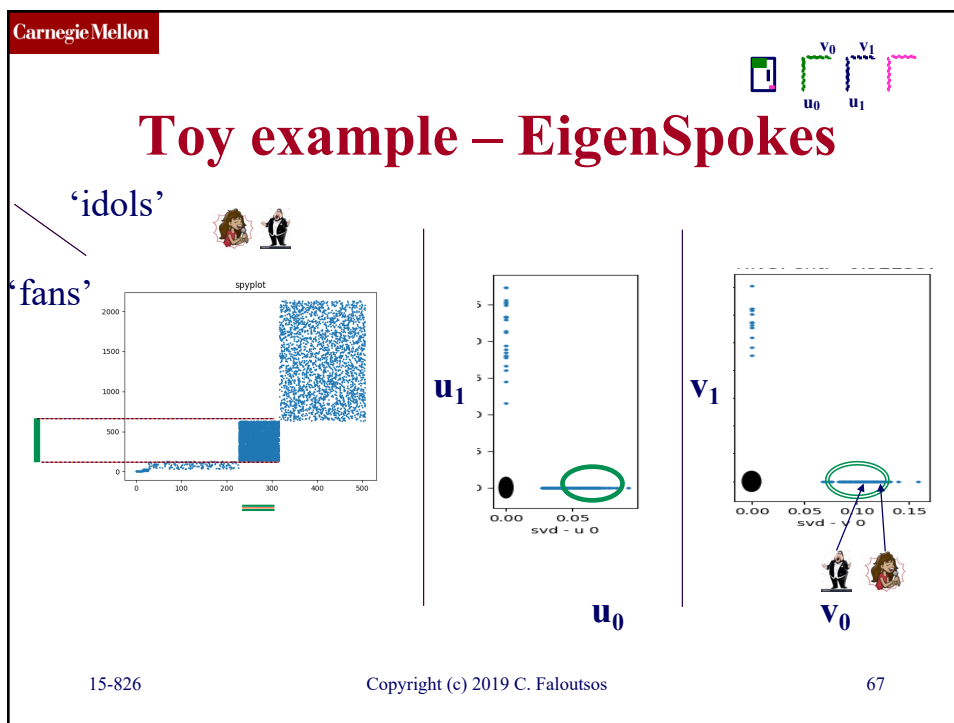
Toy example – EigenSpokes

‘idols’
‘fans’

u_1 v_1
 u_0 v_0

EigenPlots

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EigenSpokes

- Eigenvectors of adjacency matrix

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SVD - Drill

- Compute SVD by hand
 - On a block-structured, toy matrix
 - Exploiting the properties of SVD

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SVD - Drill

- Drill: find the SVD, 'by inspection' !
- Q: rank = ??

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix} \times \begin{bmatrix} & \\ & \end{bmatrix} \times \begin{bmatrix} & & \\ & & \end{bmatrix}$$

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SVD - Drill

- A: rank = 2 (2 linearly independent rows/cols)

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} | & | \\ ?? & ?? \\ | & | \end{bmatrix} \times \begin{bmatrix} ?? & 0 \\ 0 & ?? \end{bmatrix} \times \begin{bmatrix} \text{---} & ?? & \text{---} \\ \text{---} & ?? & \text{---} \end{bmatrix}$$

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SVD - Drill

- A: rank = 2 (2 linearly independent rows/cols)

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} ?? & 0 \\ 0 & ?? \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

orthogonal??

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SVD - Drill

- column vectors: are orthogonal - but not unit vectors:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & 0 \\ 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{bmatrix} \times \begin{bmatrix} ?? & 0 \\ 0 & ?? \end{bmatrix} \times \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

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SVD - Drill

- and the singular values are:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & 0 \\ 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{bmatrix} \times \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \times \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

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SVD - Drill

- Q: How to check we are correct?

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & 0 \\ 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{bmatrix} \times \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \times \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

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SVD - Drill

- A: SVD properties:
 - matrix product should give back matrix \mathbf{A}
 - matrix \mathbf{U} should be column-orthonormal, i.e., columns should be unit vectors, orthogonal to each other
 - ditto for matrix \mathbf{V}
 - matrix $\mathbf{\Lambda}$ should be diagonal, with positive values

SVD - Detailed outline

- Motivation
- Definition - properties
- Interpretation
-  • Complexity
- Case studies
- Additional properties


SVD - Complexity

- $O(n * m * m)$ or $O(n * n * m)$ (whichever is less)
- less work, if we just want singular values
- or if we want first k singular vectors
- or if the matrix is **sparse** [Berry]
- Implemented: in any linear algebra package (LINPACK, matlab, Splus/R, mathematica ...)

SVD - conclusions so far

- SVD: $A = U \Lambda V^T$: unique (*)
- **U**: document-to-concept similarities
- **V**: term-to-concept similarities
- **Λ** : strength of each concept
- dim. reduction: keep the first few strongest singular values (80-90% of 'energy')
 - SVD: picks up linear correlations
- SVD: picks up non-zero 'blobs'


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Conclusion

- How to find ‘concepts’ in a set of doc’s?
 - (~ clusters)
- SVD (= LSI) on the document-term matrix

A	U	Lambda	V ^t
N _{doc}	N _{ter}	nc	ncn
		x	x



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