



15-826: Multimedia Databases and Data Mining

Lecture #18: SVD - part I (definitions)

C. Faloutsos



Must-read Material

- [Numerical Recipes in C ch. 2.6](#);
- [Textbook Appendix D](#)



Outline

Goal: 'Find similar / interesting things'

- Intro to DB
- • Indexing - similarity search
- Data Mining



Indexing - Detailed outline

- primary key indexing
- secondary key / multi-key indexing
- spatial access methods
- fractals
- text
- Singular Value Decomposition (SVD)
- multimedia
- ...

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SVD - Detailed outline

- • Motivation
- Definition - properties
- Interpretation
- Complexity
- Case studies
- Additional properties

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SVD - Motivation

- problem #1: text - LSI: find 'concepts'
- problem #2: compression / dim. reduction

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SVD - Motivation

- problem #1: text - LSI: find ‘concepts’

document	term	data	information	retrieval	brain	lung
CS-TR1		1	1	1	0	0
CS-TR2		2	2	2	0	0
CS-TR3		1	1	1	0	0
CS-TR4		5	5	5	0	0
MED-TR1		0	0	0	2	2
MED-TR2		0	0	0	3	3
MED-TR3		0	0	0	1	1

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SVD - Motivation

- Customer-product, for recommendation system:

The diagram shows a matrix of food items (bread, lettuce, tomatoes, beef, chicken) for two groups: vegetarians (green) and meat eaters (red). Arrows point from the group names to their respective columns in the matrix.

	bread	lettuce	tomatoes	beef	chicken
vegetarians	1	1	1	0	0
	2	2	2	0	0
	1	1	1	0	0
	5	5	5	0	0
meat eaters	0	0	0	2	2
	0	0	0	3	3
	0	0	0	1	1

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SVD - Motivation

- problem #2: compress / reduce dimensionality

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Problem - specs

- $\sim 10^{**}6$ rows; $\sim 10^{**}3$ columns; no updates;
- random access to any cell(s) ; small error: OK

customer	day	Wc	Th	Fr	Sa	Su
		7/10/96	7/11/96	7/12/96	7/13/96	7/14/96
ABC Inc.		1	1	1	0	0
DEF Ltd.		2	2	2	0	0
GHI Inc.		1	1	1	0	0
KLM Co.		5	5	5	0	0
Smith		0	0	0	2	2
Johnson		0	0	0	3	3
Thompson		0	0	0	1	1

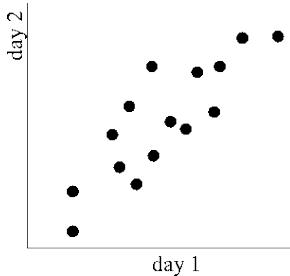
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SVD - Motivation



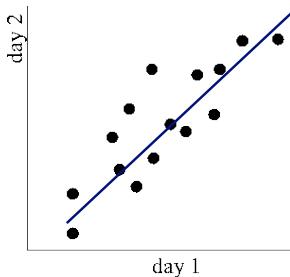
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SVD - Motivation



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SVD - Detailed outline

- Motivation
- ➡ • Definition - properties
- Interpretation
- Complexity
- Case studies
- Additional properties

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SVD - Definition

(reminder: matrix multiplication

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \times \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} ? \\ ? \end{bmatrix}$$

$3 \times 2 \quad 2 \times 1$

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SVD - Definition

(reminder: matrix multiplication

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \times \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} ? \\ ? \end{bmatrix}$$

$\overbrace{\hspace{1cm}}$ $\overbrace{\hspace{1cm}}$ $\overbrace{\hspace{1cm}}$

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SVD - Definition

(reminder: matrix multiplication

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \times \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$\xleftarrow[3 \times 2]{\quad}$ $\xrightarrow[2 \times 1]{\quad}$ $\xleftarrow[3 \times 1]{\quad}$

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SVD - Definition

(reminder: matrix multiplication

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \times \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$\xleftarrow[3 \times 2]{\quad}$ $\xrightarrow[2 \times 1]{\quad}$ $\xleftarrow[3 \times 1]{\quad}$

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SVD - Definition

(reminder: matrix multiplication

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \times \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$$

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SVD - Definition

$$\mathbf{A}_{[n \times m]} = \mathbf{U}_{[n \times r]} \boldsymbol{\Lambda}_{[r \times r]} (\mathbf{V}_{[m \times r]})^T$$

- \mathbf{A} : $n \times m$ matrix (eg., n documents, m terms)
- \mathbf{U} : $n \times r$ matrix (n documents, r concepts)
- $\boldsymbol{\Lambda}$: $r \times r$ diagonal matrix (strength of each ‘concept’) (r : rank of the matrix)
- \mathbf{V} : $m \times r$ matrix (m terms, r concepts)

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SVD - Definition

- $\mathbf{A} = \mathbf{U} \boldsymbol{\Lambda} \mathbf{V}^T$ - example:

$$\begin{array}{ccccc}
 \mathbf{A} & = & \mathbf{U} & \text{Lambda} & \mathbf{V}^T \\
 \text{Non} & & \text{Non} & \text{Non} & \text{Non} \\
 \boxed{} & - & \times & \times & \boxed{}
 \end{array}$$

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SVD - Properties

THEOREM [Press+92]: always possible to decompose matrix \mathbf{A} into $\mathbf{A} = \mathbf{U} \boldsymbol{\Lambda} \mathbf{V}^T$, where

- $\mathbf{U}, \boldsymbol{\Lambda}, \mathbf{V}$: unique (*)
- \mathbf{U}, \mathbf{V} : column orthonormal (ie., columns are unit vectors, orthogonal to each other)
 - $\mathbf{U}^T \mathbf{U} = \mathbf{I}; \mathbf{V}^T \mathbf{V} = \mathbf{I}$ (\mathbf{I} : identity matrix)
- $\boldsymbol{\Lambda}$: singular are positive, and sorted in decreasing order

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SVD - Example

- $\mathbf{A} = \mathbf{U} \Lambda \mathbf{V}^T$ - example:

$\begin{array}{c} \text{CS} \\ \downarrow \\ \text{MD} \end{array}$	$\begin{array}{l} \text{retrieval} \\ \text{data} \\ \downarrow \\ \text{inf} \quad \text{brain} \quad \text{lung} \end{array}$	$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$	$=$	$\begin{bmatrix} 0.18 & 0 \\ 0.36 & 0 \\ 0.18 & 0 \\ 0.90 & 0 \\ 0 & 0.53 \\ 0 & 0.80 \\ 0 & 0.27 \end{bmatrix}$	\times	$\begin{bmatrix} 9.64 & 0 \\ 0 & 5.29 \end{bmatrix}$	\times	$\begin{bmatrix} 0.58 & 0.58 & 0.58 & 0 & 0 \\ 0 & 0 & 0 & 0.71 & 0.71 \end{bmatrix}$
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 CMU SCS

SVD - Example

- $\mathbf{A} = \mathbf{U} \Lambda \mathbf{V}^T$ - example:

$\begin{array}{c} \text{CS} \\ \downarrow \\ \text{MD} \end{array}$	$\begin{array}{l} \text{retrieval} \\ \text{data} \\ \downarrow \\ \text{inf} \quad \text{brain} \quad \text{lung} \end{array}$	$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$	$=$	$\begin{bmatrix} 0.18 & 0 \\ 0.36 & 0 \\ 0.18 & 0 \\ 0.90 & 0 \\ 0 & 0.53 \\ 0 & 0.80 \\ 0 & 0.27 \end{bmatrix}$	\times	$\begin{bmatrix} 9.64 & 0 \\ 0 & 5.29 \end{bmatrix}$	\times	$\begin{bmatrix} 0.58 & 0.58 & 0.58 & 0 & 0 \\ 0 & 0 & 0 & 0.71 & 0.71 \end{bmatrix}$
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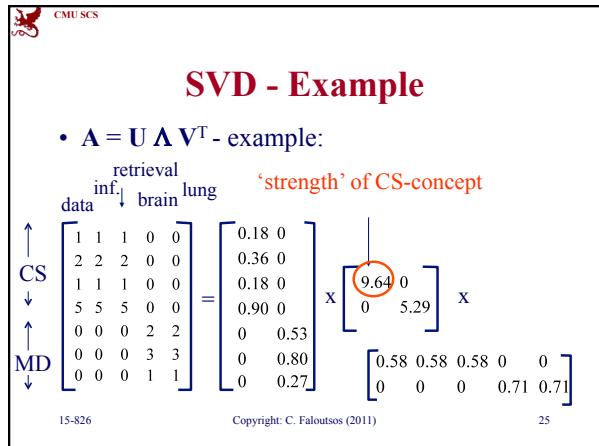
 CMU SCS

SVD - Example

- $\mathbf{A} = \mathbf{U} \Lambda \mathbf{V}^T$ - example: doc-to-concept similarity matrix

$\begin{array}{c} \text{CS} \\ \downarrow \\ \text{MD} \end{array}$	$\begin{array}{l} \text{retrieval} \\ \text{data} \\ \downarrow \\ \text{inf} \quad \text{brain} \quad \text{lung} \end{array}$	$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$	$=$	$\begin{bmatrix} 0.18 & 0 \\ 0.36 & 0 \\ 0.18 & 0 \\ 0.90 & 0 \\ 0 & 0.53 \\ 0 & 0.80 \\ 0 & 0.27 \end{bmatrix}$	\times	$\begin{bmatrix} 9.64 & 0 \\ 0 & 5.29 \end{bmatrix}$	\times	$\begin{bmatrix} 0.58 & 0.58 & 0.58 & 0 & 0 \\ 0 & 0 & 0 & 0.71 & 0.71 \end{bmatrix}$
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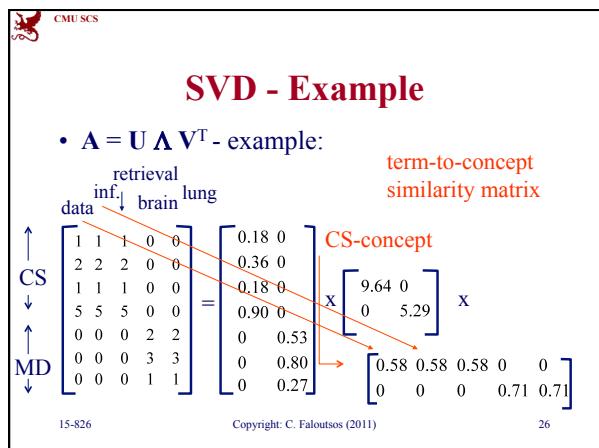
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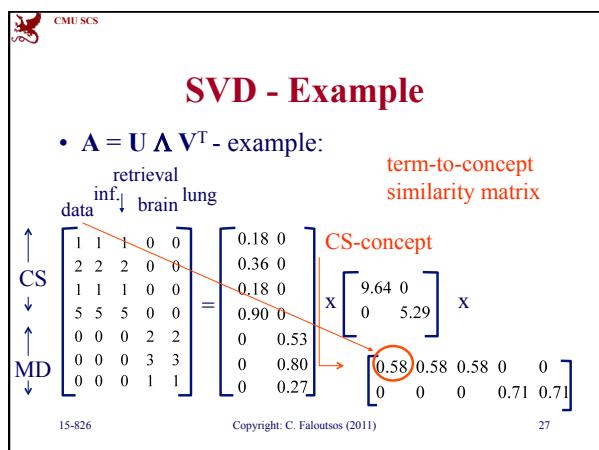
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SVD - Detailed outline

- Motivation
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- • Interpretation
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SVD - Interpretation #1

‘documents’, ‘terms’ and ‘concepts’:

- U : document-to-concept similarity matrix
- V : term-to-concept sim. matrix
- Λ : its diagonal elements: ‘strength’ of each concept

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SVD – Interpretation #1

‘documents’, ‘terms’ and ‘concepts’:

Q: if A is the document-to-term matrix, what is $A^T A$?

A:

Q: $A A^T$?

A:

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SVD – Interpretation #1

‘documents’, ‘terms’ and ‘concepts’:

Q: if A is the document-to-term matrix, what is $A^T A$?

A: term-to-term ($[m \times m]$) similarity matrix

Q: $A A^T$?

A: document-to-document ($[n \times n]$) similarity matrix

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SVD properties

- \mathbf{V} are the eigenvectors of the covariance matrix $\mathbf{A}^T \mathbf{A}$
 - \mathbf{U} are the eigenvectors of the Gram (inner-product) matrix $\mathbf{A} \mathbf{A}^T$

Further reading:

1. Ian T. Jolliffe, *Principal Component Analysis* (2nd ed), Springer, 2002.
 2. Gilbert Strang, *Linear Algebra and Its Applications* (4th ed), Brooks Cole, 2005.



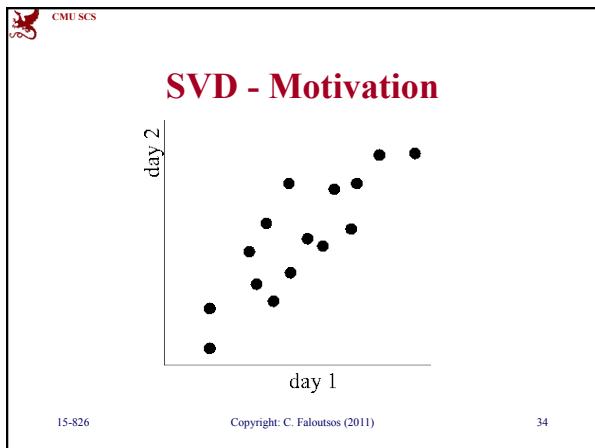
SVD - Interpretation #2

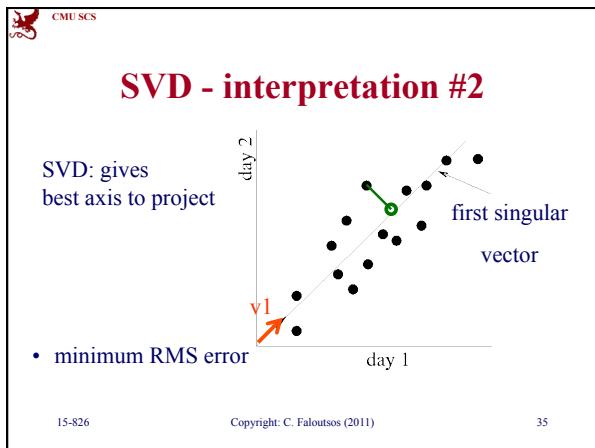
- best axis to project on: ('best' = min sum of squares of projection errors)

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SVD - Interpretation #2

customer	day	We 7/10/96	Th 7/11/96	Fr 7/12/96	Sa 7/13/96	Su 7/14/96
ABC Inc.		1	1	1	0	0
DEF Ltd.		2	2	2	0	0
GHI Inc.		1	1	1	0	0
KLM Co.		5	5	5	0	0
Smith		0	0	0	2	2
Johnson		0	0	0	3	3
Thompson		0	0	0	1	1

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SVD - Interpretation #2

- $\mathbf{A} = \mathbf{U} \Lambda \mathbf{V}^T$ - example:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0.18 & 0 \\ 0.36 & 0 \\ 0.18 & 0 \\ 0.90 & 0 \\ 0 & 0.53 \\ 0 & 0.80 \\ 0 & 0.27 \end{bmatrix} \times \begin{bmatrix} 9.64 & 0 \\ 0 & 5.29 \end{bmatrix} \times \begin{bmatrix} 0.58 & 0.58 & 0.58 & 0 & 0 \\ 0 & 0 & 0 & 0.71 & 0.71 \end{bmatrix}$$

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SVD - Interpretation #2

- $\mathbf{A} = \mathbf{U} \Lambda \mathbf{V}^T$ - example:

variance ('spread') on the v1 axis

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0.18 & 0 \\ 0.36 & 0 \\ 0.18 & 0 \\ 0.90 & 0 \\ 0 & 0.53 \\ 0 & 0.80 \\ 0 & 0.27 \end{bmatrix} \times \begin{bmatrix} 9.64 & 0 \\ 0 & 5.29 \end{bmatrix} \times \begin{bmatrix} 0.58 & 0.58 & 0.58 & 0 & 0 \\ 0 & 0 & 0 & 0.71 & 0.71 \end{bmatrix}$$

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SVD - Interpretation #2

- $\mathbf{A} = \mathbf{U} \Lambda \mathbf{V}^T$ - example:
 - $\mathbf{U} \Lambda$ gives the coordinates of the points in the projection axis

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0.18 & 0 \\ 0.36 & 0 \\ 0.18 & 0 \\ 0.90 & 0 \\ 0 & 0.53 \\ 0 & 0.80 \\ 0 & 0.27 \end{bmatrix} \times \begin{bmatrix} 9.64 & 0 \\ 0 & 5.29 \end{bmatrix} \times \begin{bmatrix} 0.58 & 0.58 & 0.58 & 0 & 0 \\ 0 & 0 & 0 & 0.71 & 0.71 \end{bmatrix}$$

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SVD - Interpretation #2

- More details
- Q: how exactly is dim. reduction done?

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0.18 & 0 \\ 0.36 & 0 \\ 0.18 & 0 \\ 0.90 & 0 \\ 0 & 0.53 \\ 0 & 0.80 \\ 0 & 0.27 \end{bmatrix} X \begin{bmatrix} 9.64 & 0 \\ 0 & 5.29 \end{bmatrix} X$$

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SVD - Interpretation #2

- More details
- Q: how exactly is dim. reduction done?
- A: set the smallest singular values to zero:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0.18 & 0 \\ 0.36 & 0 \\ 0.18 & 0 \\ 0.90 & 0 \\ 0 & 0.53 \\ 0 & 0.80 \\ 0 & 0.27 \end{bmatrix} X \begin{bmatrix} 9.64 & 0 \\ 0 & \cancel{5.29} \end{bmatrix} X$$

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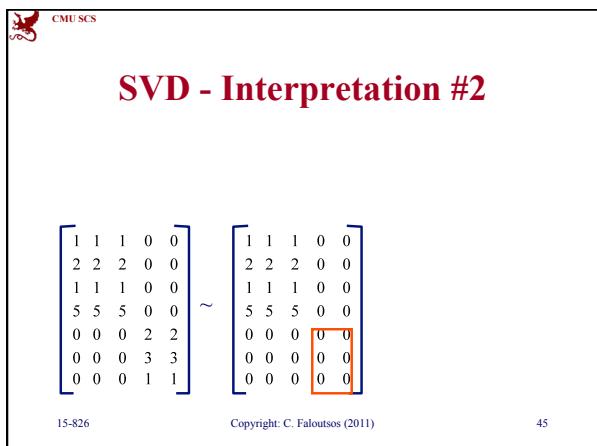
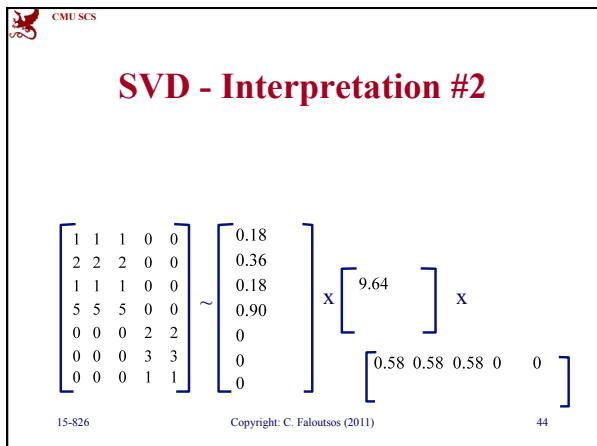
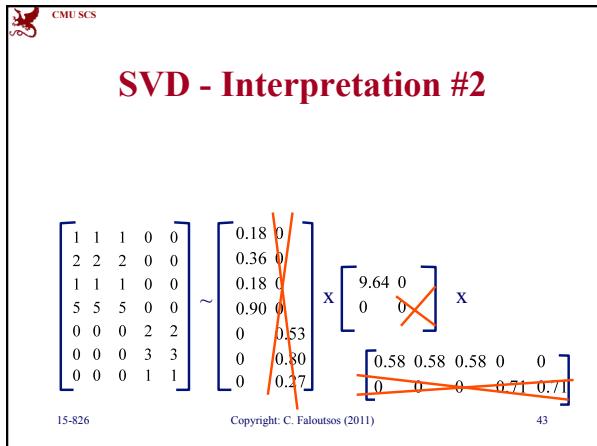
SVD - Interpretation #2

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 0.18 & 0 \\ 0.36 & 0 \\ 0.18 & 0 \\ 0.90 & 0 \\ 0 & 0.53 \\ 0 & 0.80 \\ 0 & 0.27 \end{bmatrix} X \begin{bmatrix} 9.64 & 0 \\ 0 & \cancel{5.29} \end{bmatrix} X$$

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SVD - Interpretation #2

Exactly equivalent:
‘spectral decomposition’ of the matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0.18 & 0 \\ 0.36 & 0 \\ 0.18 & 0 \\ 0.90 & 0 \\ 0 & 0.53 \\ 0 & 0.80 \\ 0 & 0.27 \end{bmatrix} \times \begin{bmatrix} 9.64 & 0 \\ 0 & 5.29 \end{bmatrix} \times \begin{bmatrix} 0.58 & 0.58 & 0.58 & 0 & 0 \\ 0 & 0 & 0 & 0.71 & 0.71 \end{bmatrix}$$

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SVD - Interpretation #2

Exactly equivalent:
 ‘spectral decomposition’ of the matrix:

$$\left[\begin{array}{cccc} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right] = \left[\begin{array}{c|c} & \\ & \\ u_1 & u_2 \\ & \\ & \end{array} \right] x \left[\begin{array}{cc} \lambda_1 & \emptyset \\ \emptyset & \lambda_2 \end{array} \right] x^{-1} \left[\begin{array}{c|c} v_1 & \\ v_2 & \end{array} \right]$$

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SVD - Interpretation #2

Exactly equivalent:
‘spectral decomposition’ of the matrix:

$$\begin{array}{c} \text{n} \\ \uparrow \\ \left[\begin{array}{ccccc} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right] \end{array} = \lambda_1 u_1 v_1^T + \lambda_2 u_2 v_2^T + \dots$$

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SVD - Interpretation #2

Exactly equivalent:
 ‘spectral decomposition’ of the matrix:

$$\begin{matrix} \text{n} & \xrightarrow{\quad m \quad} \\ \uparrow & \left[\begin{array}{ccccc} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right] & = & \xleftarrow{\quad r \text{ terms} \quad} \\ \downarrow & & & \lambda_1 u_1 v^T_1 + \lambda_2 u_2 v^T_2 + \dots \end{matrix}$$

$n \times 1 \qquad 1 \times m$

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SVD - Interpretation #2

approximation / dim. reduction:
by keeping the first few terms (Q: how many?)

$$\begin{array}{c} \uparrow \\ n \end{array} \left[\begin{array}{ccccc} m & & & & \\ \hline 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right] = \lambda_1 u_1 v^T_1 + \lambda_2 u_2 v^T_2 + \dots$$

assume: $\lambda_1 >= \lambda_2 >= \dots$

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SVD - Interpretation #2

A (heuristic - [Fukunaga]): keep 80-90% of ‘energy’ (= sum of squares of λ_i ’s)

$$\begin{array}{c} \text{m} \\ \uparrow \\ \left[\begin{array}{ccccc} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right] = \lambda_1 u_1 v^T + \lambda_2 u_2 v^T_2 + \dots \\ \text{n} \end{array}$$

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SVD - Detailed outline

- Motivation
 - Definition - properties
 - Interpretation
 - #1: documents/terms/concepts
 - #2: dim. reduction
 - #3: picking non-zero, rectangular ‘blobs’
 - Complexity
 - Case studies
 - Additional properties

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SVD - Interpretation #3

- finds non-zero ‘blobs’ in a data matrix

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0.18 & 0 \\ 0.36 & 0 \\ 0.18 & 0 \\ 0.90 & 0 \\ 0 & 0.53 \\ 0 & 0.80 \\ 0 & 0.27 \end{bmatrix} x \begin{bmatrix} 9.64 & 0 \\ 0 & 5.29 \end{bmatrix} x$$

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SVD - Interpretation #3

- finds non-zero ‘blobs’ in a data matrix

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ \hline 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right] = \left[\begin{array}{cc} 0.18 & 0 \\ 0.36 & 0 \\ 0.18 & 0 \\ 0.90 & 0 \\ \hline 0 & 0.53 \\ 0 & 0.80 \\ 0 & 0.27 \end{array} \right] \times \left[\begin{array}{cc} 9.64 & 0 \\ 0 & 5.29 \end{array} \right] \times \left[\begin{array}{cccccc} 0.58 & 0.58 & 0.58 & 0 & 0 \\ 0 & 0 & 0 & 0.71 & 0.71 \end{array} \right]$$

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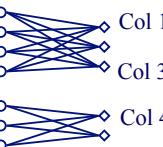
54

 CMU SCS

SVD - Interpretation #3

- finds non-zero ‘blobs’ in a data matrix =
- ‘communities’ (bi-partite cores, here)

1	1	1	0	0
2	2	2	0	0
1	1	1	0	0
5	5	5	0	0
0	0	0	2	2
0	0	0	3	3
0	0	0	1	1

Row 1 

Row 4 

Row 5 

Row 7 

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SVD - Interpretation #3

- Drill: find the SVD, ‘by inspection’!
- Q: rank = ??

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} ?? \\ ?? \\ ?? \\ ?? \\ ?? \end{bmatrix} \times \begin{bmatrix} ?? & 0 \\ 0 & ?? \end{bmatrix} \times \begin{bmatrix} ?? \\ ?? \end{bmatrix}$$

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SVD - Interpretation #3

- A: rank = 2 (2 linearly independent rows/cols)

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} | & | & | \\ ?? & ?? & ?? \\ | & | & | \end{bmatrix} \times \begin{bmatrix} ?? & 0 \\ 0 & ?? \end{bmatrix} \times \begin{bmatrix} --- & ?? & --- \\ --- & ?? & --- \end{bmatrix}$$

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SVD - Interpretation #3

- A: rank = 2 (2 linearly independent rows/cols)

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} ?? & 0 \\ 0 & ?? \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

orthogonal??

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SVD - Interpretation #3

- column vectors: are orthogonal - but not unit vectors:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & 0 \\ 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{bmatrix} \times \begin{bmatrix} ?? & 0 \\ 0 & ?? \end{bmatrix} \times \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

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SVD - Interpretation #3

- and the singular values are:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & 0 \\ 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{bmatrix} \times \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \times \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

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SVD - Interpretation #3

- Q: How to check we are correct?

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & 0 \\ 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{bmatrix} \times \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \times \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

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SVD - Interpretation #3

- A: SVD properties:

- matrix product should give back matrix **A**
- matrix **U** should be column-orthonormal, i.e., columns should be unit vectors, orthogonal to each other
- ditto for matrix **V**
- matrix **A** should be diagonal, with positive values

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SVD - Detailed outline

- Motivation
- Definition - properties
- Interpretation
- Complexity
- Case studies
- Additional properties



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SVD - Complexity

- $O(n * m * m)$ or $O(n * n * m)$ (whichever is less)
- less work, if we just want singular values
 - or if we want first k singular vectors
 - or if the matrix is sparse [Berry]
- Implemented: in any linear algebra package (LINPACK, matlab, Splus/R, mathematica ...)

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SVD - conclusions so far

- SVD: $A = U \Lambda V^T$: unique (*)
- U : document-to-concept similarities
- V : term-to-concept similarities
- Λ : strength of each concept
- dim. reduction: keep the first few strongest singular values (80-90% of ‘energy’)
 - SVD: picks up linear correlations
 - SVD: picks up non-zero ‘blobs’

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