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15-826: Multimedia Databases and Data Mining

Lecture #13: Power laws
Potential causes and explanations
C. Faloutsos

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Must-read Material

- Power laws, Pareto distributions and Zipf's law Contemporary Physics 46, 323-351 (2005)

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Optional Material

- (optional, but very useful: Manfred Schroeder *Fractals, Chaos, Power Laws: Minutes from an Infinite Paradise* W.H. Freeman and Company, 1991) – ch. 15.

Outline

Goal: 'Find similar / interesting things'

- Intro to DB
- Indexing - similarity search
- Data Mining

Indexing - Detailed outline

- primary key indexing
- secondary key / multi-key indexing
- spatial access methods
  - z-ordering
  - R-trees
  - misc
- fractals
  - intro
  - applications
- text
- ...

Indexing - Detailed outline

- fractals
  - intro
  - applications
    - disk accesses for R-trees (range queries)
    - dimensionality reduction
    - selectivity in M-trees
    - dim. curse revisited
    - "fat fractals"
    - quad-tree analysis [Gaede+]
    - nn queries [Belussi+]

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This presentation

- Definitions
- Examples and counter-examples
- Generative mechanisms

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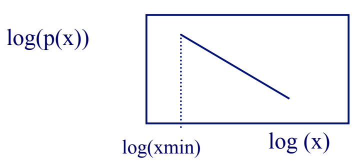
Definition

- p(x) = C x ^ (-a) (x >= xmin)
- Eg., prob(city pop. between x + dx)



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For discrete variables

$$p_k = C k^{-a} \qquad (k > 0)$$

Or, the Yule distribution:

$$p_k = C\, B(k,a)$$

$$B(k,a) = \Gamma(k)\Gamma(a) / \Gamma(k+a) \approx k^{-a}$$

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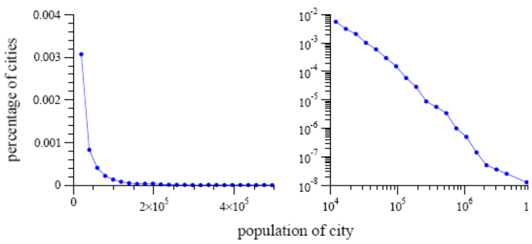
[Newman, 2005]



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Estimation for *a*

$$a = 1 + n\left[\sum_{i=1}^{n} \ln(x_i / x_{\min})\right]^{-1}$$

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Examples

- Word frequencies
- Citations of scientific papers
- Web hits
- Copies of books sold
- Magnitude of earthquakes
- Diameter of moon craters
- …

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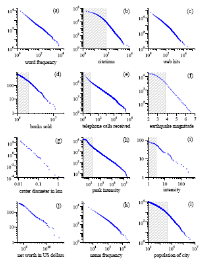
[Newman 2005]



Rank-frequency plots
Or Cumulative D.F.

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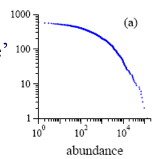
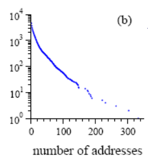
NOT following P.L.

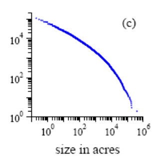
'abundance' of species





Number of addresses

Cumul. D.F.



Size of forest fires

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This presentation

- Definitions
- Examples and counter-examples
- Generative mechanisms
  - Combination of exponentials
  - Inverse
  - Random walk
  - Yule distribution = CRP
  - Percolation
  - Self-organized criticality
  - Other

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Combination of exponentials

Let $p(y) = e^{ay}$

- eg., radioactive decay, with half-life –a
- (= collection of people, playing russian roulette)

Let $x \sim e^{by}$

- (every time a person survives, we double his capital)

$p(x) = p(y)*dy/dx = 1/b\ x^{(-1+a/b)}$

- Ie, the final capital of each person follows P.L.

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Combination of exponentials

- Monkey on a typewriter:
- *m*=26 letters equiprobable;
- space bar has prob. q_s
- Freq(x-th most frequent word) = $x^{(-a)}$

see Eq. 47 of [Newman]:

$a = [2 \ln(m) - \ln(1 - q_s)] / [\ln m - \ln(1 - q_s)]$

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Inverses of quantities

- y follows $p(y)$ and goes through zero
- $x = 1/y$
- Then $p(x) = \ldots = - p(y) / x^2$
- For $y \sim 0$, x has power law tail.

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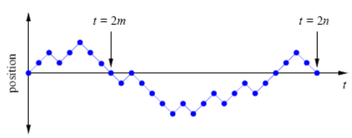
Random walks



Inter-arrival times PDF: *p(t)* ~ ??

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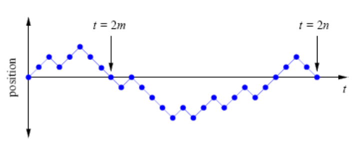
Random walks



Inter-arrival times PDF: $p(t) \sim t^{-3/2}$

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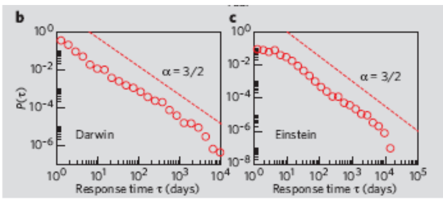
Random walks

J. G. Oliveira & A.-L. Barabási Human Dynamics: The Correspondence Patterns of Darwin and Einstein. *Nature* **437,** 1251 (2005) . [PDF]



1 **Figure 1 | The correspondence patterns of Darwin and Einstein.** 22

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Yule distribution and CRP

Chinese Restaurant Process (CRP):

Newcomer to a restaurant

- Joins an existing table (preferring large groups
- Or starts a new table/group of its own, with prob *1/m*

a.k.a.: rich get richer; Yule process

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Yule distribution and CRP

Then:

Prob(k people in a group) = p_k

$= (1 + 1/m) B(k, 2+1/m)$

$\sim k^{-(2+1/m)}$

(since B(a,b) ~ a ** (-b) : power law tail)

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Yule distribution and CRP

- Yule process
- Gibrat principle
- Matthew effect
- Cumulative advantage
- Preferential attachement
- 'rich get richer'

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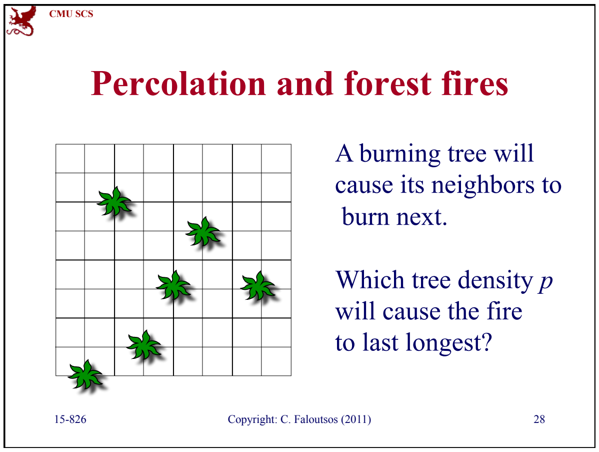
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Percolation and forest fires
A burning tree will cause its neighbors to burn next.
Which tree density p will cause the fire to last longest?
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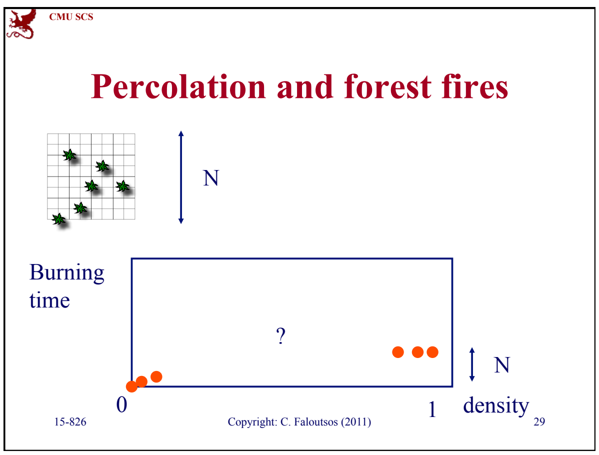
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Percolation and forest fires
N
Burning time
?
N
0
1
density
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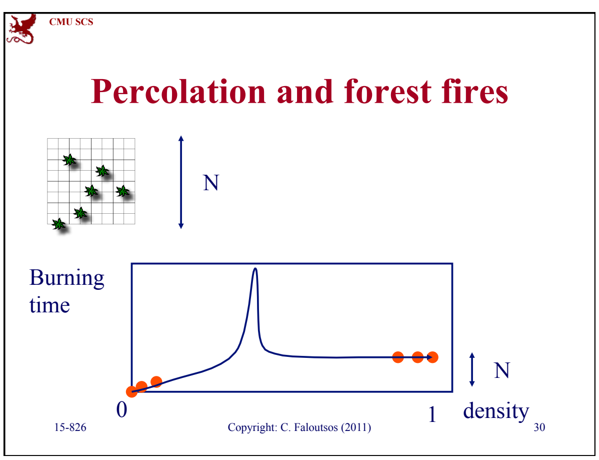
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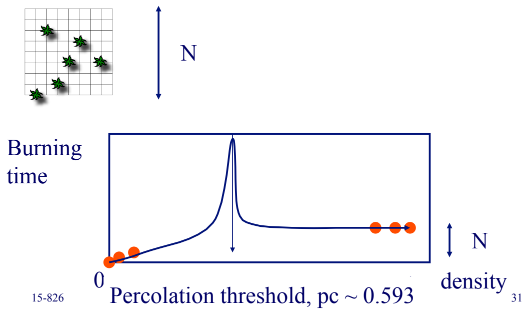
Percolation and forest fires



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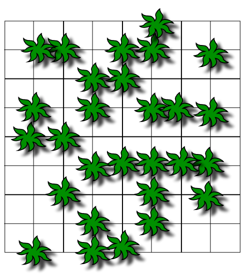
Percolation and forest fires



At pc ~ 0.593:
No characteristic scale;
'patches' of all sizes;
Korcak-like 'law'.

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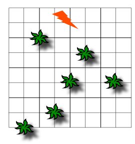
Self-organized criticality

- Trees appear at random (eg., seeds, by the wind)
- Fires start at random (eg., lightning)
- Q1: What is the distribution of size of forest fires?



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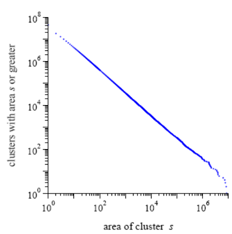
Self-organized criticality

- A1: Power law-like

CCDF



Area of cluster s

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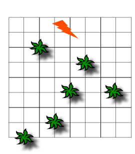
Self-organized criticality

- Trees appear at random (eg., seeds, by the wind)
- Fires start at random (eg., lightning)
- Q2: what is the average density?



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Self-organized criticality

- A2: the critical density pc ~ 0.593

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Self-organized criticality

- [Bak]: size of avalanches ~ power law:
- Drop a grain randomly on a grid
- It causes an avalanche if height(x,y) is >1 higher than its four neighbors

[Per Bak: *How Nature works*, 1996]

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Other

- Random multiplication
- Fragmentation

-> lead to lognormals (~ look like power laws)

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Others

Random multiplication:

- Start with C dollars; put in bank
- Random interest rate s(t) each year t
- Each year t: C(t) = C(t-1) * (1+ s(t))

- Log(C(t)) = log(C) + log(..) + log(..) … -> Gaussian

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Others

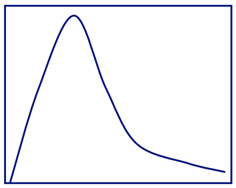
Random multiplication:

- Log(C(t)) = log(C) + log(..) + log(..) … -> Gaussian

- Thus C(t) = exp(Gaussian)
- By definition, this is Lognormal

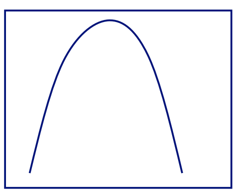
Others

Lognormal:



Others

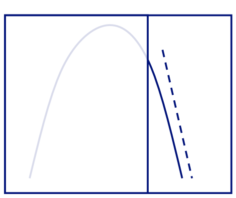
Lognormal:



parabola

Others

Lognormal:



parabola

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Other

- Random multiplication
- ➡ Fragmentation

-> lead to lognormals (~ look like power laws)

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Other

- Stick of length 1
- Break it at a random point x (0<x<1)
- Break each of the pieces at random

- Resulting distribution: lognormal (why?)

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Conclusions

- Power laws and power-law like distributions appear often
- (fractals/self similarity -> power laws)
- Exponentiation/inversion
- Yule process / CRP / rich get richer
- Criticality/percolation/phase transitions
- Fragmentation -> lognormal ~ P.L.