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One-Handed Juggling: A Dynamical Approach to a Rhythmic Movement Task

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ABSTRACT. The skill of rhythmic juggling a ball on a racket was investigated from the viewpoint of nonlinear dynamics. The difference equations that model the dynamical system were analyzed by means of local and nonlocal stability analyses. These analyses showed that the task dynamics offer an economical juggling pattern that is stable even for open-loop actuator motion. For this pattern, two types of predictions were extracted: (a) Stable periodic bouncing is sufficiently characterized by a negative acceleration of the racket at the moment of impact with the ball, and (b) a nonlinear scaling relation maps different juggling trajectories onto one topologically equivalent dynamical system. The relevance of these results for the human control of action was evaluated in an experiment in which subjects ($N = 6$) performed a comparable task of juggling a ball on a paddle. Task manipulations involved different juggling heights and gravity conditions of the ball. The following predictions were confirmed: (a) For stable rhythmic performance, the paddle's acceleration at impact is negative and fluctuations of the impact acceleration follow predictions from global stability analysis; and (b) for each subject, the realizations of juggling for the different experimental conditions are related by the scaling relation. These results permit one to conclude that humans reliably exploit the stable solutions inherent to the dynamics of the given task and do not overrule these dynamics by other control mechanisms. The dynamical scaling serves as an efficient principle for generating different movement realizations from only a few parameter changes and is discussed as a dynamical formalization of the principle of motor equivalence.

Key words: juggling, motor control, motor equivalence, nonlinear dynamics, stability analysis

The skill of juggling has attracted the fascination of people for centuries. It is, from the viewpoint of motor control, a complex multiple-degree-of-freedom movement that requires the control of one or two hands with respect to several balls or other manipulanda. Our goal in this study was to identify the laws and rules responsible for the achievement of the coordinated perception and action that are involved in such tasks.

In research on human movement coordination and robotics, two theoretical perspectives can be distinguished. Motor program theory, which is rooted in a control theoretic approach, has dominated research on motor control. In this approach, one typically uses feedback and feedforward control signals to overrule any inherent dynamics and impose desired movements or dynamics. An alternative approach, based on a dynamical systems perspective, has developed over the last decade and applies the mathematical tools and concepts from nonlinear dynamical systems theory. This approach stresses the need for and opportunity of taking advantage of existing task dynamics rather than canceling them. The second approach was explored in the present study, in which a model system was proposed for the task of bouncing a ball on a racket, a simple kind of juggling. We have used theoretical predictions from the model and empirical results from human performance as examples to evaluate the relevance of dynamical systems theory. Moreover, these results suggest a new perspective on Bernstein's notion of motor equivalence, in the spirit of dynamical system theory.

The skill of juggling has been addressed by several investigations from the control theoretic and motor program perspective. The overall goal of these investigations has been to establish algorithms and programs that are capable of the control and execution of motor actions. The essence of this perspective is that control is completely attributed to the central command level and execution is relegated to the effector system, which is logically separate from the command level. Control is exerted by means of a combination of feedback and feedforward signals. Feedback control sig-

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nals are computed on the basis of sensor measurements and provide flexibility and compensatory adaptability to environmental changes. Feedforward signals are generated at a separate planning level and stored in internal representations. (For more details about control theoretic and motor program approaches, see Brady, Hollerbach, Johnson, Lozano-Perez, & Mason, 1982; Schmidt, 1975; Shapiro & Schmidt, 1982.) This stance has been taken by Austin (1976), who analyzed human ball juggling by structuring the recurrent pattern of hand movements and the flying balls into a sequence of motor subroutines that can be planned and successively executed. Aboaf, Drucker, and Atkeson (1989) studied a robot's juggling action as a robot arm bounced one ball on a planar surface. A learning algorithm was developed in which a task controller used the predicted landing location of the ball to choose the correct parameters of the algorithm for hitting the ball. In both of these examples, emphasis was put on the planning of the actuator's trajectory and stabilization was obtained by a feedback controller.

A slightly different route was taken by Bühler and Koditschek (1990) and Rizzi and Koditschek (1992), who developed a nonlinear algorithm that controlled the movements of a robot arm whose goal was to bounce a ball rhythmically at a fixed height. The spatiotemporal path of the actuator was specified to be a scaled mirror image of the ball's flight trajectory. The new element in this "mirror algorithm" was that the movements of ball and paddle were tightly coupled at every moment in time and essentially could be viewed as two strongly coupled nonlinear oscillators (Koditschek, 1993). Thus, a bidirectional influence between controller and controlled object was established.

One motivation for the present work was the dynamical systems approach advocated by Kugler, Kelso, and Turvey (1980, 1982) and Schöner, Haken, and Kelso (1986). The concept of bidirectionality, or mutuality between the actor and the manipulated object, is a fundamental component of their version of a dynamical approach to action control. The actor is viewed as a participant in a dynamical regime and not as a controller in a hierarchical regime. Human behavior is modeled as a nonlinear dynamical system, whose space–time evolution displays features observed and studied in models of self-organizing physical systems. Saltzman and Kelso (1987) proposed a task dynamical framework in which the constraints for the articulators' movement are derived from a functionally defined task space. The stable performance of movements within given task constraints as well as the flexible adaptation to changing conditions become intrinsic properties of the dynamical system. Thus, the desired kinematic trajectory of an effector requires neither a detailed plan nor any contingency or replanning procedures for dealing with unexpected perturbations. This perspective has received theoretical and empirical support in a number of investigations (e.g., Kelso, 1981; Kelso, Tuller, Vatikiotis-Bateson, & Fowler, 1984; Schöner, Haken, & Kelso, 1986; Sternad, Turvey, & Schmidt, 1992; Turvey, 1990). This conceptual framework has been applied to humans performing the skill of juggling three to seven balls (Beek, 1989; Beek & van Santvoord, 1992). In this work, fundamental concepts of dynamical systems theory, such as circle map dynamics, provided the tools for accounting for the temporal and spatial patterning of balls and hand loop times.

The movement task investigated here is a simple juggling task, paddle juggling, in which one keeps a ball in the air by hitting it upward with a planar surface. The ball bounces on the paddle because of the elastic impact, and its trajectory in the air follows the standard laws of ballistic flight. The first mathematical treatment of a paddle-juggling-like task by Wood and Byrne (1981) focused on a system consisting of one ball bouncing on a planar surface that periodically moves in the vertical direction. High-frequency motions relevant for industrial problems, in which the surface's amplitude is much smaller than the object's amplitude (as, for example, vibration in gear boxes), were investigated in this analysis. Assuming a perodical movement of the surface implies the simplification that no influence is exerted from the ball to the effector and, consequently, the effector acts as an open-loop system. The bouncing-ball system received further attention by Holmes (1982) and Guckenheimer and Holmes (1983), who showed that a ball bouncing on a periodically driven planar surface exhibits steady states, period bifurcations, strange attractors, and chaotic motion (see also Tufillaro, Abbott, & Reilly, 1992). Their analysis, however, was still confined to the special case of a bouncing ball on a vibrating table; that is, the amplitude of the table was relatively small when compared with the amplitude of the ball's flight trajectory, and, as a consequence, the table's amplitude could still be neglected.

When investigating human juggling, however, one must direct the emphasis of an investigation upon the paddle's movement and, hence, can no longer neglect its amplitude. In identifying the relevant components for modeling human juggling, it is helpful conceptually to differentiate the paddle-juggling system into three subsystems: the ball, the effector (the musculoskeletal system and the paddle), and the perceptual coupling between the ball and the effector. The ball's dynamics are strictly defined by the laws of ballistic flight and are coupled to the paddle by the laws of elastic impact. The movement of the effector can be sufficiently captured by rigid-body dynamics. Because these two subsystems can be reasonably well approximated by established physical principles, the question remains: How complex a perceptual coupling is required for a successful performance of the task? In this light, for instance, mirror law control (Bühler & Koditschek, 1990) can be interpreted as a very strong coupling that forces the effector to mirror the motion of the ball. In contrast, the analysis of Guckenheimer and Holmes (1983) assumes unidirectional coupling, for it is only the surface that drives the ball, whereas the ball has no effect on the surface's motion. Coupling from ball to paddle—bidirectional coupling, as could be established by perception (e.g., Schöner, 1991)—was not included in their formal model. Therefore, their analysis can be interpreted as a way to investigate whether and to what degree one requires additional perceptual coupling to maintain a stable juggling pattern.

If the unidirectionally coupled system, the open-loop system, is able to obtain stability, a human may exploit this "passive" stability, which is inherent to the dynamics of the task. Importantly, if humans do exploit the passive stability, their control system must be sensitive to task dynamical properties. On the one hand, this sensitivity could come from a planning instance. Solutions based on movement plans (and the corresponding feedback stabilization), however, are unlikely to make use of passive stability, unless the planner has full understanding of the physical properties of the system, which requires a quite complex planner. An alternative solution is that the dynamics of paddle and ball merely get attracted into a passively stable regime. Within this regime, perceptual coupling may contribute little to the control of the task, and the system could be governed by its open-loop stability properties. Note that this does not imply that the movement is executed in an open-loop fashion, but, rather, that the open-loop stability of the system will be the essential component of stable performance and dominate the observed movement behavior.

Our motivation in the present work was to investigate whether a dynamical analysis of the paddle-juggling system, in which the paddle executes a periodic movement with only unidirectional coupling (i.e., no perceptual coupling), as in the analysis of Guckenheimer and Holmes (1983), can provide a basis for modeling the relevant aspects of the hand–ball coordination of human paddle juggling. Within this goal, a twofold route was taken. First, we modified Guckenheimer and Holmes' dynamical analysis of the bouncing-ball system to serve as a model for a movement system that can perform stable juggling of one ball on a paddle. Second, we conducted an experiment in which human subjects performed a one-handed paddle-juggling task similar to the ball-bouncing model. The hypotheses derived from the analyses allow a qualitative and quantitative comparison between analytical and experimental results and, hence, an evaluation of the biological validity of such dynamical models.

Dynamical Analysis of Paddle Juggling

To formulate a model for paddle juggling, one must capture the motions of ball and paddle in two state vectors, each describing position and velocity of ball and paddle, respectively (Figure 1):

$$\mathbf{x}_B = (x_B, \dot{x}_B)^T, \quad \mathbf{x}_P = (x_P, \dot{x}_P)^T. \tag{1}$$

The motion is confined to the vertical dimension, where a positive sign defines the upward direction. According to Liouville's theorem, the system must dissipate energy to display asymptotic stability (Lichtenberg & Lieberman, 1982). This energy loss is captured by the coefficient of restitution, $\alpha \in [0,1]$, characterizing the elastic impact:

$$(\dot{x}_B^+ - \dot{x}_P) = -\alpha(\dot{x}_B^- - \dot{x}_P). \tag{2}$$

The variables $\dot{x}_B^-$ and $\dot{x}_B^+$ denote the ball's velocity immediately before (–) and immediately after (+) the impact. The paddle's mass is assumed to be sufficiently larger than the ball's mass so that the effects of the impact on the paddle's trajectory are negligible. To render the analysis tractable, one reduces the two continuous dynamical equations of motion for ball and paddle by discretizing at the point immediately before the impact between ball and paddle. At this moment, both positions, x_B and x_P, are identical and collapse into one state, thus reducing the dimensionality of the sys-

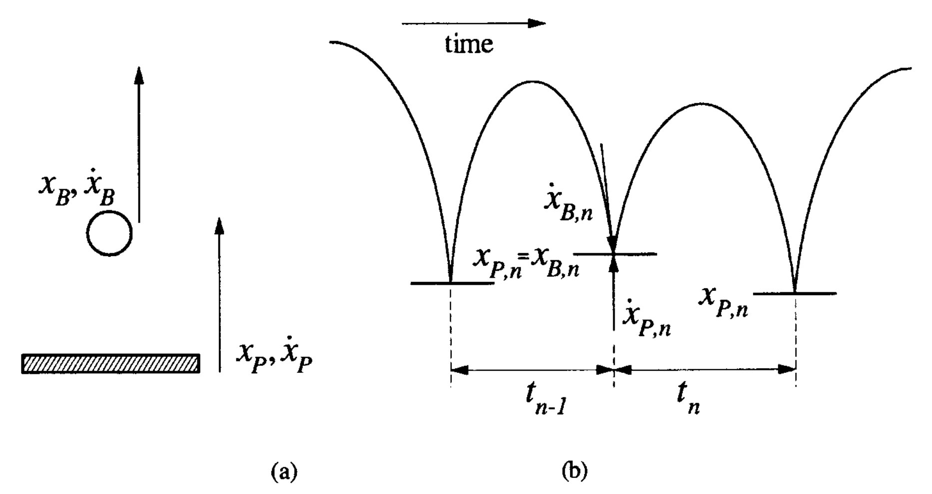

FIGURE 1. (a) Sketch of the paddle-juggling model. **(b)** Notation for the discretization.

tem by one. This discretization is equivalent to taking the Poincaré section, $\Sigma = \{(\mathbf{x}_B, \mathbf{x}_P) \in \Re^4 | x_B - x_P = 0\}$. Information about the stability properties of the continuous ball-bouncing system is completely contained in the recurrent pattern of the discrete points of impact.

In the new discrete notation $x_{P,n}$ refers to the corresponding vertical position of the paddle at the nth impact, $\dot{x}_{P,n}$ to the paddle velocity, and $\dot{x}_{B,n}$ to the ball's velocity immediately before the nth impact. From the equations for the ballistic flight and the elastic impact, discrete equations can be written for the ball as

$$\dot{x}_{B,n+1} = -\sqrt{[(1+\alpha)\dot{x}_{B,n}]^2 - 2g(x_{P,n+1} - x_{P,n})}; \qquad (3)$$
$$x_{B,n+1} = x_{P,n+1};$$

where $x_{P,n+1} = x_{P,n+1}(t_n)$, and t_n results from

$$-0.5gt_n^2 + [(1+\alpha)\dot{x}_{P,n} - \alpha\dot{x}_{B,n}]t_n - (x_{P,n+1} - x_{P,n}) = 0.$$

It should be noted that these equations include the term $x_{P,n+1}$ on the right side, which is the position of the paddle at the $(n+1)$th impact. Evidently, without explicit knowledge of the paddle's dynamics Equation 3 is not solvable. Even if $x_{P,n+1}$ is the result of a simple sinusoidal trajectory of the paddle, the third equation is transcendental and cannot be solved analytically for the time to impact, t_n. Nevertheless, one can perform a number of dynamical analyses to gain insight into some of the system's characteristic properties. In particular, stable cycles manifest themselves in fixed points of Equation 3. The understanding of the existence and nature of these fixed points will provide explicit hypotheses for the subsequent experiment with human subjects. The detailed analytical derivations can be found in Schaal (1996), and the results are summarized in the following paragraphs.

Characteristics of Stable Fixed Points

For any arbitrary, smooth periodic paddle motion, the dynamical system (Equation 3) has at least one asymptotically stable fixed point, provided that the peak velocity of the paddle exceeds a minimum value to compensate for the energy lost during the elastic impact. Formally, this can be expressed by three conditions, which have to be fulfilled:

$$\text{i. } t_n = \tau, \text{ where } \tau = \text{constant.}$$
$$\text{ii. } \dot{x}_{P,n} = \frac{\dot{x}_{B,n}(\alpha - 1)}{(\alpha + 1)} = 0.5g\tau\frac{(1-\alpha)}{1+\alpha} > 0. \qquad (4)$$
$$\text{iii. } -2g\frac{(1+\alpha^2)}{(1+\alpha)^2} < \ddot{x}_{P,n} < 0.$$

Condition (i) ensures periodicity with constant period τ, and condition (ii) states that the paddle must be moving in the positive direction at the impact, that is, upward toward the ball. It is Condition (iii) that provides the most interesting information about the equilibrium dynamics. Dynamically stable fixed points with period one can be achieved only if the paddle is in its decelerating phase at impact and the magnitude of deceleration is bounded by (iii).

Figure 2 illustrates the influence of the paddle's acceleration at impact, by means of a numerical simulation. In each of the three time series, 25 balls start at the same position with slightly different initial velocities and go through a series of 12 bounces. In Figure 2a, each ball is hit with a positively accelerating movement, indicated by the upward concavity of the paddle's trajectory. This leads to a quick divergence of the balls' trajectories, indicating that the solution is unstable. Paddle trajectories with constant velocity, as shown in Figure 2b, exhibit neutral stability; different initial conditions lead to different ball trajectories, and perturbations are not compensated for and persist over time. Only the decelerating paddle movement (Figure 2c) stabilizes the balls' trajectories, leading to the asymptotic convergence of the discrete point of impact to a stable fixed point.

It should be pointed out that it is not obvious that a movement system should start decelerating before hitting the ball. Related work in robotics, for instance, focused on solutions (a) and (b), and stable juggling could therefore be obtained only in connection with feedback controllers (Aboaf et al., 1989; Bühler, 1990; Bühler & Koditschek, 1990; Rizzi & Koditschek, 1992). Solution (c), which exploits the dynamic stability intrinsic to the task, was not taken advantage of by such algorithms. In the light of the earlier discussion, the hypothesis is that it is this nonobvious solution (c) that a movement system should tune into for stable, but simultaneously parsimonious control.

Local Linear Stability

Local linear stability analysis gives a first assessment of the stability properties of the fixed points determined for the model system (Equation 3). Linearizing about an equilibrium point results in a matrix equation for the ball,

$$\mathbf{x}_{B,n+1} = \mathbf{A}\mathbf{x}_{B,n}. \qquad (5)$$

The characteristic 2×2 matrix $\mathbf{A}$ has two eigenvalues λ_1, λ_2 The condition for stable equilibrium points in discrete systems is that the absolute value of both eigenvalues must lie in the interval [0, 1]. It therefore suffices to test the larger absolute eigenvalue, $|\lambda_{max}|$, for this condition and distinguish among the following three cases within the range of $\ddot{x}_{P,n}$ specified in Equation (4[iii]):

$$\text{i. For } 0 > \ddot{x}_{P,n} \geq -g\frac{(1-\alpha)^2}{(1+\alpha)^2}: \quad 1 > |\lambda_{max}| \geq \alpha.$$
$$\text{ii. For } -g\frac{(1-\alpha)^2}{(1+\alpha)^2} > \ddot{x}_{P,n} > -g: \quad |\lambda_1 = \lambda_2 = \lambda_{max}| = \alpha. \qquad (6)$$
$$\text{iii. For } -g \geq \ddot{x}_{P,n} > -2g\frac{1+\alpha^2}{(1+\alpha)^2}: \quad 1 > |\lambda_{max}| \geq \alpha.$$

The equations show that local stability depends only on the paddle's acceleration at impact, $\ddot{x}_{P,n}$; the coefficient of restitution, α; and the gravitational constant, g. Because α and

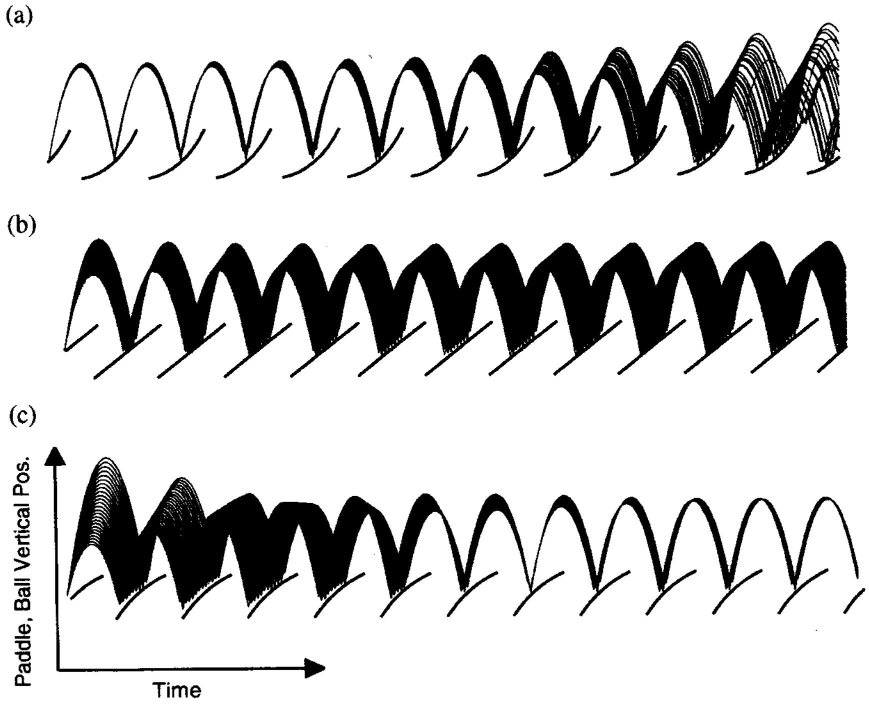

FIGURE 2. Simulation of the influence of shape of paddle trajectory on stability. Twenty-five balls start on the left at the same position but with slightly different initial velocity conditions, and each ball is hit with a hit trajectory having (**a**) positive acceleration, (**b**) constant velocity, and (**c**) negative acceleration.

g are constant and not under the control of an effector system, $\ddot{x}_{P,n}$ can serve as the main variable for the assessment of different juggling solutions in the experiment. For the analytical evaluation of local stability, the range of $\ddot{x}_{P,n}$, where $|\lambda_{max}|$ is at a minimum, is of primary importance. The results are illustrated in Figure 3, which shows $|\lambda_{max}|$ against $\ddot{x}_{P,n}$. In the central range, with its limits specified in Equation (6[ii]), $|\lambda_{max}|$ has a minimum, which is a plateau over the whole interval of (6[ii]). This result provides a first analytical constraint on possible solutions. Differentiating between the stability within this range of acceleration values requires a global stability analysis.

Topological Orbital Equivalence

Obviously, when humans perform the model task, they will produce not one, but a family of solutions. The question is whether all successful but, nevertheless, different solutions can be normalized such that quantitative comparisons are possible. Mathematically, this question is addressed by topological orbital equivalence (TOE), which tests whether one dynamical system can be continuously transformed into another one. A formal way of establishing TOE is to find an orientation-preserving homeomorphism between two dynamical systems (Arnol'd, 1983; Jackson, 1989). The following scaling relation,

$$h := \begin{cases} \dot{x}'_{B,n} = c\dot{x}_{B,n} \\ \dot{x}'_{P,n} = c\dot{x}_{P,n} \\ \ddot{x}'_{P,n} = c^2 x_{P,n} \quad c > 0, \\ t'_n = ct_n \quad (\Rightarrow \tau' = c\tau) \end{cases} \tag{7}$$

fulfills the requirements of TOE for Equation 3. For any constant, c, the primed variables also fulfill Equation 3, which can be verified by inserting them into these equations (Schaal, 1996). This implies that by choosing $c = 1/t_n$, each periodic paddle-juggling system is normalized by h to unit period without changing its dynamical properties. Hence, because of h, any further analysis of paddle juggling can be performed on one system with unit period. For the present analyses, it is important that the scaling relation does not affect the acceleration of the paddle. If the paddle trajectories are transformable into each other by this scaling, then their acceleration at impact remains invariant. Thus, acceleration at impact can directly serve as a measure of local stability (Figure 3) and will form an important criterion for the empirical analysis.

Figure 4 illustrates the scaling relation when applied to an idealized sinusoidal paddle trajectory. Three trajectories with three different amplitudes and three periods, respectively, are portrayed in the phase plane. These trajectories

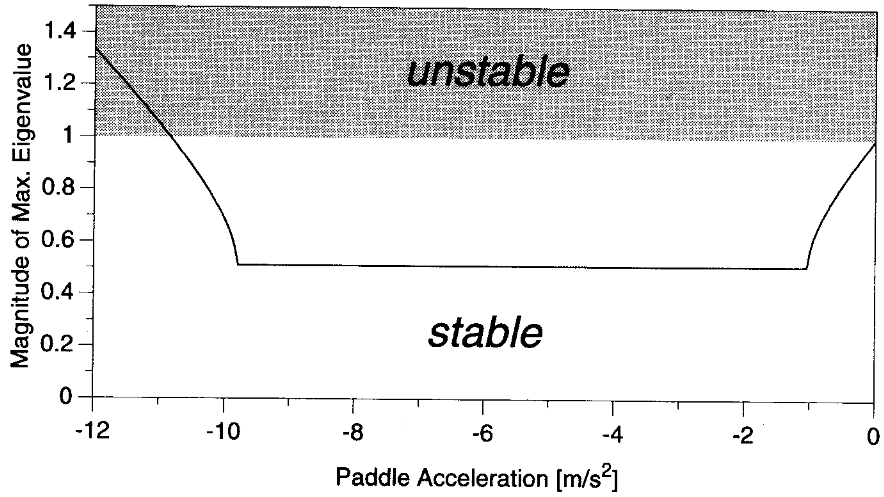

FIGURE 3. Absolute magnitude of the larger eigenvalue, $|\lambda_{max}|$, as a function of paddle acceleration at impact. The region $|\lambda_{max}| \in [0,1]$ (white) gives stable solutions of the dynamical system; $|\lambda_{max}| > 1$ (gray) indicates unstable solutions.

are transformable into each other by h and are therefore dynamically equivalent. Lines are drawn that connect points of identical acceleration across ellipses of a continuous range of periods and amplitudes. Note that these isoacceleration lines are not straight, even though the dynamical properties of the three systems are identical. Therefore, descriptions that are anchored in the coordinate system of the phase plane cannot reveal the invariances under the scaling relation h.

It is important that h holds not only for the discretized measures, but also for continuous measures. With a view toward analyzing experimental data, two dependent measures of the continuous movement, the amplitude of the paddle's displacement and of its velocity, are of interest and are defined as follows: $x_{P,amp} = x_{P,\max} - x_{P,\min}$, $\dot{x}_{P,amp} = \dot{x}_{P,\max} - \dot{x}_{P,\min}$. When examining Equation 7 with respect to how position and velocity scale with period, it can be seen that $x_{P,amp}$ and $\dot{x}_{P,amp}$ are straightforwardly related to period by the following invariant ratios:

$$\frac{x'_{P,amp}}{\tau'^2} = \frac{c^2 x_{P,amp}}{(c\tau)^2} = \frac{x_{P,amp}}{\tau^2} = constant,$$
$$\frac{\dot{x}'_{P,amp}}{\tau'} = \frac{c\dot{x}_{P,amp}}{c\tau} = \frac{\dot{x}_{P,amp}}{\tau} = constant, \tag{8}$$

which means that the scaling relation h requires that $x_{P,amp}$ be proportional to the square of paddle period, τ^2, and that $\dot{x}_{P,amp}$ be proportional to τ. These dependent measures can easily be extracted from kinematic data of human performance. It is of interest to determine whether humans scale their trajectories in a manner such that different trajectories with different periods and amplitudes obey this nonlinear relationship. With respect to motor control, this implies that a whole range of task-specific trajectories form a class of actions that are unified by this nonlinear scaling relation. This can be interpreted as an economizing principle in the control of action, such that one class of movements can be parameterized to fulfill different variants of the required task. It is also important to point out that this scaling of movements is independent of the stability of the solution. Trajectories that are scaled with respect to each other do not necessarily have to produce a stable juggling rhythm, and,

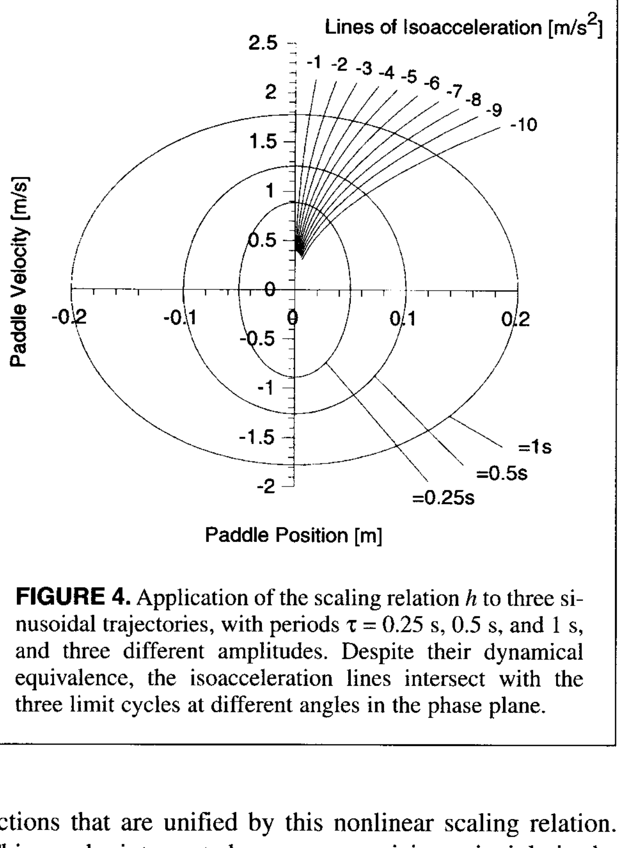

FIGURE 4. Application of the scaling relation h to three sinusoidal trajectories, with periods $\tau = 0.25$ s, 0.5 s, and 1 s, and three different amplitudes. Despite their dynamical equivalence, the isoacceleration lines intersect with the three limit cycles at different angles in the phase plane.

conversely, stable solutions do not have to follow the scaling relation.

Nonlocal Stability

Without making any assumption about the shape of the paddle trajectory, the preceding analyses established that paddle acceleration at impact can serve as a measure of stability, that this criterion has to satisfy the specified necessary conditions, and that a scaling relation exists that preserves the dynamic properties of the juggling system. Results of local stability analysis, however, do not differentiate between conditions for a wide range of the criterion measure, $\ddot{x}_{P,n}$. Intuitively, one would expect that the stability properties of a solution, $\ddot{x}_{P,n}$, change as $\ddot{x}_{P,n}$ changes: A very large acceleration, $\ddot{x}_{P,n}$, implies a very quickly changing paddle trajectory, and its basin of attraction should differ from that of a trajectory that has a relatively small $\ddot{x}_{P,n}$. These questions can be addressed by a nonlocal stability analysis. The most common method for performing such nonlocal stability of an equilibrium point of a dynamical system is by finding a Lyapunov function, which is a potential function of the state variables and which is formulated to have a global minimum at this equilibrium point. If the time derivative of this potential function is always negative, meaning that its value monotonically decreases with time, then the system converges to the minimum of the Lyapunov function. Because, by definition, the minimum is the equilibrium point, global stability of the system is proved. For a nonlinear system, a Lyapunov function candidate can be derived from the linearized system. The candidate function, L_n, for the linearized dynamical system (Equation 5) is (e.g., Chen, 1984)

$$L_n = \mathbf{x}_{B,n}^T \mathbf{P} \mathbf{x}_{B,n}. \tag{9}$$

Negative time derivatives can be obtained if the matrix **P** satisfies the equation

$$\mathbf{A}^T\mathbf{P}\mathbf{A} - \mathbf{P} = -\mathbf{I}. \tag{10}$$

A is the system matrix of Equation 5, and I is the identity matrix. For the discretized sysem, the value of L_n must continuously decrease when $\mathbf{x}_{B,n}$ is recursively iterated through Equation 3. Thus, a ΔL can be defined between two successive impacts n and $n + 1$ of ball and paddle:

$$\Delta L = L_{n+1} - L_n = \mathbf{x}_{B,n+1}^T \mathbf{P} \mathbf{x}_{B,n+1} - \mathbf{x}_{B,n}^T \mathbf{P} \mathbf{x}_{B,n}, \tag{11}$$

where the nonlinear system Equation 3 must be inserted for $\mathbf{x}_{B,n+1}$. For any state $\mathbf{x}_{B,n}$, ΔL may serve as a measure of how quickly the ball will converge to the stable equilibrium point. Whereas negative values of ΔL indicate that $\mathbf{x}_{B,n}$ lies in the basin of attraction, a single positive ΔL characterizes $\mathbf{x}_{B,n}$ as unstable.

Because the Lyapunov function was derived for the linearized system, its range as a stability measure is restricted to a subset of the state space around the equilibrium point and is therefore, strictly speaking, not global, but only nonlocal. By using numerical optimization analysis, one can assess these stability properties by simulating the dynamics of the paddle-juggling system given by Equation 3. At time $t = 0$ an equilibrium point was defined to be at the impact position, $x_P = 0$, and the juggling period was set to $\tau = 0.4$ s. The scaling relation h ensures that these values can be chosen arbitrarily without losing generality of the results. The locally relevant section of the paddle trajectory around the equilibrium point was modeled as a sixth-order polynomial in time (we empirically determined the order six to provide sufficient accuracy for the given purpose):

$$x_p(t) = c_0 + c_1t + c_2t^2 + c_3t^3 + c_4t^4 + c_5t^5 + c_6t^6. \tag{12}$$

For the given impact conditions, $x_p(t = 0) = 0$, c_0 must be zero. The constant c_1 is also determined, because the paddle velocity at impact is fully determined by the ballistic flight and the coefficient of restitution. At impact, the second derivative of Equation 12, $\ddot{x}_p(t = 0) = 2c_2$. This acceleration was set to 19 different values, taken from the range of local stability [–.5, –9.5]. Our goal for the optimization was to adjust the constants c_3 to c_6 for each of the 19 $\ddot{x}_p(t = 0)$ so that the largest and steepest basin of attraction for the equilibrium point could be achieved. Values of ΔL were calculated by starting the ball at 2,500 different initial conditions in the vicinity of the equilibrium point. The sum of all 2,500 ΔLs for a given set of parameters, $\Sigma\Delta L$, was defined as an operational measure quantifying stability for each $\ddot{x}_p(t = 0)$. The ball's initial conditions were determined by different deviations from the impact time, $t = 0$, and impact velocities around the equilibrium point of $\ddot{x}_P$. We chose the range of the initial values to cover an appropriately large neighborhood around the equilibrium point, but, as our purpose for performing this calculation was to get relative evaluations of $\ddot{x}_P(t)$, the actual range limits could be chosen freely: $t_{init} \in [-0.18\tau, +0.18\tau]$, $\dot{x}_{B,init} \in [-4 \text{ m/s}^2, -1 \text{ m/s}^2]$. The initial conditions were obtained by discretizing the intervals into 50 values each. The optimization was performed with Powell's conjugate gradient method (Press, Flannery, Teukolsky, & Vetterling, 1989).

Figure 5 shows the numerical results of $\Sigma\Delta L$ as a function of $\ddot{x}_P(t)$. Note that small $\Sigma\Delta L$ corresponded to high global stability. Because we optimized the trajectory of the paddle corresponding to each of the different $\ddot{x}_P(t)$ to obtain maximal stability, the results express the best possible case for each $\ddot{x}_P(t)$. As expected, a relative differentiation of stability within the range of locally stable $\ddot{x}_P(t)$ was achieved. The optimal solutions of the juggling task are performed when $\ddot{x}_{P,n} \in [-3.5 \text{ m/s}^2, -1.5 \text{ m/s}^2]$. It should be emphasized that this result, like all the previous results, is generally valid for any arbitrary smooth periodic paddle trajectory.

Stability is closely related to variability, because weakly stable states are accompanied by larger fluctuations than highly stable states are, and have longer relaxation times when perturbed (Kelso, Schöner, Scholz, & Haken, 1987). Therefore, the variability of $\ddot{x}_{P,n}$ should increase in proportion to the numerical estimate of the global stability index, $\Sigma\Delta L$. For the empirical evaluation, this relative increase in

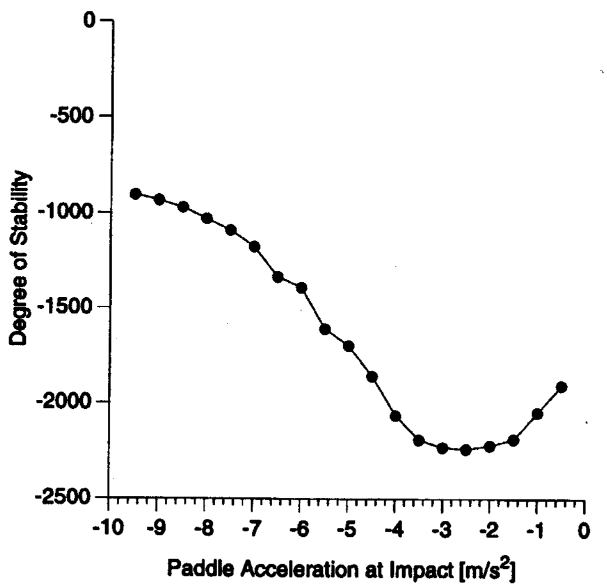

FIGURE 5. Global stability, as function of paddle acceleration at impact.

fluctuations will be operationalized in the standard deviations of $\ddot{x}_{P,n}$.

Predictions for the Experiment

On the basis of the analyses above, a juggling pattern can be characterized and evaluated in terms of its stability properties. It has been argued that this movement, when unidirectionally coupled to the ball, becomes a system that displays passive stability without the need for additional regulation with the help of visual perception. An alternative solution to the task, however, may be active control of the paddle on the basis of some kind of explicit or implicit planning strategy that can impose qualitatively different stable solutions by means of feedback control through visual perception (Aboaf et al., 1989; Bühler, 1990; Bühler & Koditschek, 1990). These two ways of control and their combinations span the range of solutions that human subjects could possibly pursue when trying to juggle a ball. If the solution follows the one defined by the open-loop dynamics of the task, the analyses outlined above make the following predictions:

1. At the moment of impact between ball and paddle, the trajectory of the paddle is in its decelerative phase, that is, $\ddot{x}_{P,n}$ has a negative value. The range is specified by the actual values of constants in the experiment.
2. The variability of $\ddot{x}_{P,n}$ should be proportional to the numerical estimate of the global stability index, $\Sigma\Delta L$.
3. The scaling relation h requires that $x_{P,amp}$ be proportional to the square of paddle period, τ^2, and that $\dot{x}_{P,amp}$ be proportional to τ, if the paddle acceleration at impact is held invariant.

Method

Subjects

Six subjects volunteered for the experiment (2 were women, 4 were men). They were graduate and postgraduate students at the Massachusetts Institute of Technology and were, on average, 33.4 years old. Five subjects were right-handed, one was left-handed; the dominant hand performed the juggling movement.

Apparatus

To allow a direct comparison of the formal analysis with the actual movement execution, we constructed an apparatus that simulated the model task as closely as possible. Figure 6 is a sketch of a subject juggling one ball in the vertical direction. The paddle was mounted to a pantograph linkage of 1.0 m length, whose two hinges were connected to a stanchion of height 1.0 m. The paddle was a commercially available "Koosh paddle," which consisted of a circular frame of diameter 0.30 m, covered by an elastic fabric. The coefficient of restitution, α, was experimentally determined to be 0.71. The pantograph linkage was a lever arm with a parallelogram-like arrangement such that the paddle's surface stayed horizontal during its movement. The distal end of the linkage had an attached handle, which the subjects held in a fully pronated grip, and downward movement of the subjects' hand produced upward movement of the paddle. Subjects were positioned behind the handle so that the linkage was horizontally aligned with their forearm and their elbow joint was at an approximately right angle. Although the pantograph's motion was curvilinear, the effective movements of the linkage, both at the hand and at the paddle, could be considered vertical, because the amplitudes generally did not leave the linear range (±30°).

A table tennis ball of diameter 0.03 m was fixed to an aluminum boom of length 1 m, which, in turn, was attached to a stanchion by a hinge joint at a height of 1.2 m. At the boom's opposite end, a weight could be affixed, which modified the flight properties of the ball. In a first-order approximation, this corresponded to a change of gravity for the ball's ballistic flight. Because of the rigid linkage of the ball to the boom, the trajectory of the ball described a curvilinear path. At high juggling amplitudes, subjects exceeded the linear range of this motion, which introduced a nonlinearity in the trajectory of the ball. However, it was numerically determined that parabolic curves still fitted the ball's trajectory with high significance (r^2 = .99), and the ball's trajectory was thus treated as a ballistic flight in the vertical dimension. The boom and paddle were aligned so that, in the horizontal position, the ball rested on the center of the paddle's surface. A potentiometer, attached to the boom's hinge, measured the angular displacement of the boom; we used this measurement to determine the ball's vertical displacement. Similarly, the collection of the position data of the actuator's movement was done by a high-resolution position encoder that was attached in one of the hinge joints at the stanchion.

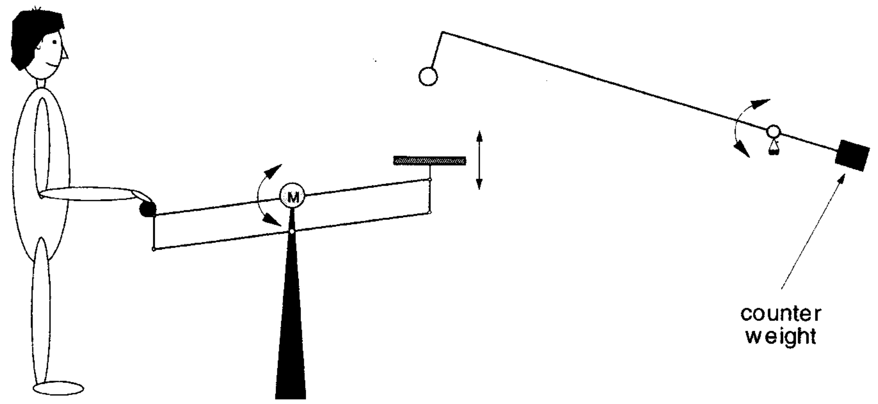

FIGURE 6. Apparatus and task in the experiment.

Data Collection and Data Reduction

Three MVME 147 computers in a VME bus collected and processed the data at a sampling frequency of 1000 Hz. Velocity data of ball and paddle were derived on-line by numerical differentiation of the position data and subsequent filtering, by using a third-order, low-pass Butterworth filter with a cut-off frequency of 30 Hz. Because of memory limitations, only every 10th data point was stored, which corresponds to downsampling of the data to 100 Hz. Filtering the data on-line at 1000 Hz resulted in a higher bandwidth than off-line filtering of the downsampled data. Delays introduced by the purely forward filters were negligible because of the high sampling frequency. We used a high-pass, second-order, Butterworth filter to filter the position data to eliminate slow drifting of the average paddle position (cut-off frequency 0.5 Hz, zero-delay filter). We numerically differentiated the filtered velocity data to derive accelerations for ball and paddle.

The point of impact was defined to be at the time at which the ball's displacement was at a minimum within each cycle. The period of the paddle was determined as the interval between two successive zero crossings of acceleration at which the paddle velocity was positive. Once the times of the impacts were obtained, the corresponding measures of position, velocity, and acceleration for the paddle's and ball's movements could be calculated. All position data were taken with respect to the absolute coordinate system of the pantograph linkage. Absolute values, however, were irrelevant, because only relative measures were entered into the analysis. For the computation of amplitude measures of the paddle, the data were converted into the coordinate system of the phase portrait, $x,\dot{x}$, and subsequently converted into its polar coordinates θ,r. The position data were normalized such that the mean position coincided with the origin. Dividing the range of the angular polar coordinate into 400 equally spaced values allowed calculation of the intersection of a straight line at each of these angles with the paddle's trajectory in the phase portrait. The values of all intersections at each angle were averaged, and we used the mean and its standard deviation to determine the mean phase portrait of every trial. Subsequently, the amplitudes of position and velocity, $x_{P,amp}$ and $\dot{x}_{P,amp}$, were computed as the difference between maxima and minima of the position and velocity data in the mean phase plot.

We averaged all discrete measures characterizing the impact, that is, position, velocity, and acceleration at the point of impact, across all cycles for each trial (between 20 and 34 cycles) to obtain mean values per trial. Subject mean values were computed as the arithmetic mean of the two trials per condition.

Design

A two-factor design was chosen, with three different juggling amplitudes (high, medium, and low) and two different ball gravity properties (normal and reduced). Medium amplitude was defined to be the height at which each subject preferred to juggle the ball. High amplitude was defined as a high but still comfortable range. Low amplitude was defined as a low as possible amplitude that did not let the ball merely vibrate on the paddle and, thus, still required active control over the ball. A control condition was included in which the subject moved the paddle rhythmically at a comfortable frequency and amplitude without bouncing a ball. The second factor in the design, the gravity conditions for the ball, were manipulated by attaching a counterweight at the distal end of the boom (Figure 6). The chosen weight reduced the gravity constant to $g = 7.0$ m/s^2. When no weight was attached, the gravity condition was normal: $g = 9.81$ m/s^2. Two trials were performed for each of the three amplitude conditions, the control condition, and the two gravity conditions of the ball. The total of 16 trials was grouped into two blocks. Within each block, the ball's gravity was held constant and the four amplitude conditions (including the control condition) were randomized. The order of blocks was counterbalanced for the 6 subjects.

Experimental Procedure and Instructions

Before the experiment, subjects were instructed about the task. Subjects were given about 5 min to practice, which was sufficient to make them feel comfortable with the task. During this practice, they were asked to explore different amplitudes and frequencies so that they would find three distinguishable heights where they could juggle the ball stably and comfortably. The experimenter also pointed out that for the high amplitudes, the juggling height should still stay below the angle at which the boom's angular excursion became increasingly curvilinear. At the beginning of the data collection, the subjects stood right behind the pantograph, grasping the handle from the top in a pronated grip. Before each trial, the subject was informed about the particular amplitude condition. Data collection started when the rhythmic movement was stable. If the subject failed to maintain a steady rhythm, the trial was repeated. Each trial lasted 30 s. After the first block of eight trials, the ball's gravity was changed and the subject practiced again for 5 min, until he or she adjusted to the new flight behavior of the ball. The whole experiment lasted approximately 30 min. Learning effects during the practicing were not examined.

Results

Kinematic Description of the Movements of Paddle and Ball

Figure 7a gives a first qualitative impression of the coordination of paddle and ball in a 5-s window of a typical trial in which a medium ball amplitude juggle was performed under normal gravity conditions. The time series shows that the paddle moved in an approximately sinusoidal waveform. The height of the ball's trajectory showed slight variations, but this did not seem to perturb the overall coordinative pattern. For instance, after the slightly lower apex of the fourth bounce, the average height was regained without any problem. Importantly, the moment of contact was unambiguously in the upward branch of the paddle's trajectory, where acceleration takes on a negative value. Figure 7b shows the phase portrait of an entire trial; the dots on the limit cycle mark the points of contact between ball and paddle. In Figure 7c, the central band represents the mean trajectory, the inner and outer bands illustrate its 95% confidence interval. The dot corresponds to the mean impact point during this trial.

To test that the experimental conditions actually produced different solutions and that the subjects actually performed the juggling task at three different ranges of ball amplitudes, we subjected subject means of the ball's amplitude, $x_{B,\,amp}$, to a 3 × 2 analysis of variance (ANOVA), with the two variables, juggling height (low, medium, high) and gravity (normal, reduced). The control condition was excluded. The analysis yielded a significant interaction, $F(1, 5) = 31.68$; $p < .0001$, a significant main effect for juggling height, $F(2, 10) = 97.89$; $p < .0001$, and for gravity, $F(1, 5) = 44.52$; $p < .001$. This main effect of juggling height verified that subjects fulfilled the task and juggled the ball at three different ranges of amplitudes (Figure 8). When gravity was reduced, the subjects juggled the ball at a higher amplitude, even at low juggling amplitudes where the difference between the two gravity conditions reached statistical significance in a pairwise *t* test ($p < .05$). The significant interaction, together with the second main effect of gravity, implies that gravity additionally influenced the height of the ball's trajectory. Figure 8 displays the mean values of all subjects' performance as a function of the categorical task.

The amplitudes of the paddle's movement, $x_{P,\,amp}$, were analyzed by performing the same 3 × 2 ANOVA; the control trials were again excluded. The results for $x_{P,\,amp}$ showed a significant main effect, $F(2, 10) = 52.99$; $p < .0001$, which indicated that bouncing the ball at three different heights was achieved by three different amplitudes of the paddle's movement. Neither the interaction nor the second main effect for gravity was significant. The paddle movement's period was completely dependent on the ball amplitude, and an analysis was therefore redundant. The individual subjects' means for the paddle's amplitude in the six experimental conditions are summarized in Table 1.

Acceleration at Impact

As we discussed in the dynamical analysis of the model system, the stability of a bouncing pattern can be determined by the acceleration of the paddle at the point of contact with the ball, $\ddot{x}_{P,\,n}$. To test whether the subjects preferred particular acceleration values and whether the task requirements had a differential effect on $\ddot{x}_{P,\,n}$, we conducted a 3 × 2 ANOVA, with the variables juggling height (low, medium, high) and gravity (normal, reduced). As in the preceding analyses, the control trials were of no concern and were thus excluded from the analysis. None of the results of this ANOVA reached significance. Hence, neither juggling height nor gravity led to statistically significant differences in the $\ddot{x}_{P,\,n}$. In other words, the acceleration with which the end effector hit the ball was indifferent to the variations specified in the task conditions. Looking at the absolute values of $\ddot{x}_{P,\,n}$ reveals that subjects stayed within the range specified as stable in the local stability analysis. By inserting the explicit experimental values for the coefficient of restitution and gravity in Equation 6, the quantitative limits of the range of local stability could be determined to be $\ddot{x}_{P,\,n} \in [0.0\ \text{m/s}^2, -10.09\ \text{m/s}^2]$ for G_{normal} and $\ddot{x}_{P,\,n} \in [0.0\ \text{m/s}^2, -7.20\ \text{m/s}^2]$ for $G_{reduced}$.

Figure 9 gives an overview of the trial means of all conditions of the individual subjects. Subjects clustered around individually confined ranges that differed for each subject. The first observation was tested by means of regression analyses in which $\ddot{x}_{P,\,n}$ was regressed against juggling height and gravity for each individual subject. Neither of the six regressions yielded significant dependencies. Second, to test whether the cluster size for the different gravity conditions per subject in Figure 9 changed as a function of gravity, we pooled the $\ddot{x}_{P,\,n}$ of the different juggling heights per

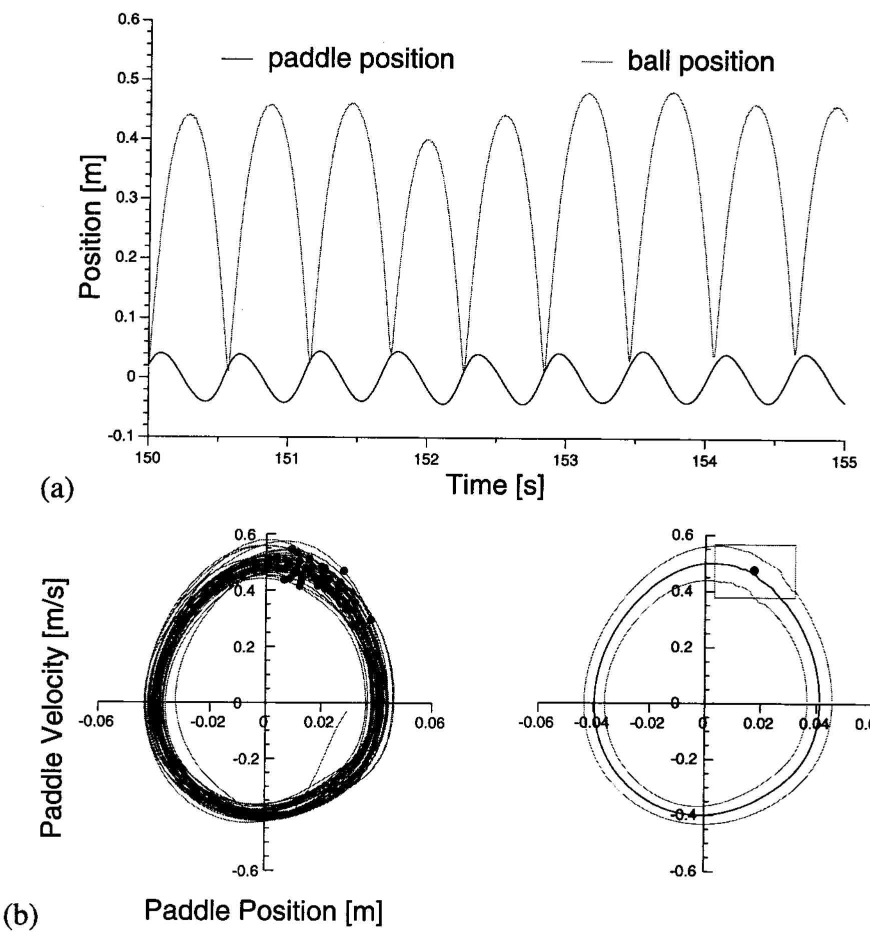

FIGURE 7. (**a**) A 5-s window of a time series showing position of ball and paddle for the juggling movement of one typical subject; (**b**) phase plot of the entire trial, where the black dots denote the points of impact between paddle and ball; (**c**) mean phase plot with standard error bands for the trajectories and the standard error in x and $\dot{x}$ for the impact points.

gravity condition for each subject and used pairwise t tests to compare their standard deviations. These tests yielded no significance. Third, we conducted a one-way analysis of variance on the trial means of the 6 subjects, to test for differences between the clusters of the individual subjects. The ANOVA revealed a significant effect, $F(5, 55) = 93.73$, $p < .0001$. Pairwise post hoc Tukey tests specified these differences: Although Subject 1 was identical to Subjects 5 and 6, and Subject 2 was identical to Subject 3, all other comparisons showed significant differences ($p < .05$). Because there was no significant difference between the two gravity conditions across subjects, the means of $\ddot{x}_{P,n}$ of all 6 subjects across all six conditions were pooled. The six values ranged between -8.75 m/s^2 and $-.50$ m/s^2; the mean value across all conditions and all 6 subjects was -3.44 m/s^2. As is evident from Figure 9, the trial means stayed in the range that is analytically designated as stable for G_{normal} and $G_{reduced}$, respectively, with the obvious exception of Subject 4 and Subject 5. Subject 4 was outside the limits, with two trials performed under normal gravity conditions and five trials with reduced gravity conditions, but Subject 5 was in the unstable range, with three trials at G_{normal} and two trials at $G_{reduced}$; in the Discussion section we will return to the issue of how these subjects' behavior can be explained. These trials are expected to display a very high variability in the juggling pattern, and in $\ddot{x}_{P,n}$ in particular. Eighty-one percent of the total number of trials were performed in the stable range. This result supports Prediction 1, which required that $\ddot{x}_{P,n}$ (a) be negative and (b) lie within the specified limits. It shows that subjects, indeed in most cases, find the passively stable solution that is offered by the dynamics of the task.

Variability of Acceleration at Impact

Prediction 2 concerned the variability that is associated with solutions of different nonlocal stability. To answer the question whether less stable solutions were accompanied by higher variability, we analyzed the standard deviations of the acceleration at impact, $SD\ddot{x}_{P,n}$. The same 3×2

ANOVA, with the variables juggling height (low, medium, high) and gravity (normal, reduced), was conducted on $SD\ddot{x}_{P,n}$. As for $\ddot{x}_{P,n}$, no significant differences were found between the experimental conditions. The mean value of $SD\ddot{x}_{P,n}$ across all experimental conditions for all subjects was 2.98 m/s^2.

As the task was relatively unconstrained and subjects were free to select their own preferred ranges as their three juggling amplitudes, $x_{B,amp}$, values showed considerable scatter across subjects within the three categories. This overlap across $x_{B,amp}$ ranges yielded a relatively good distribution of values, which enabled us to use a continuous regression model. A linear regression was performed on $\ddot{x}_{P,n}$ and $SD\ddot{x}_{P,n}$ against $x_{B,amp}$ per subject and per gravity condition. It confirmed the nonsignificance of the categorical analysis by showing flat distributions across the continuous range of ball amplitudes. These negative statistical results gave evidence that the task's inherent dynamical properties seemed to override the experimental manipulations and, again, show that subjects performed the juggling task uninfluenced by the different variants resulting from the task manipulations. This can be understood as support for the analytically derived Prediction 1, that $\ddot{x}_{P,n}$ is a criterion for stable performance of the juggling task.

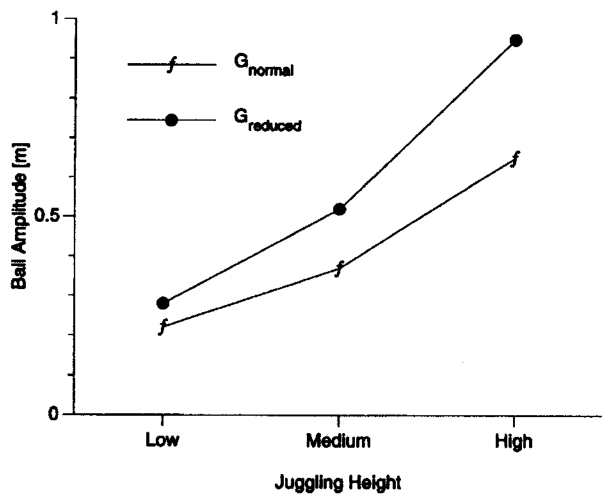

FIGURE 8. Subject means of ball amplitudes, $x_{B,amp}$, for the three juggling heights in the two gravity conditions G_{normal} and $G_{reduced}$.

As addressed above, we were interested in knowing to what degree the values of $\ddot{x}_{P,n}$ are accompanied by fluctuations. The numerical analysis on nonlocal stability allows stability predictions, provided that the juggling pattern is stable and periodic. Nonlocal stability, as indexed by $\Sigma\Delta L$, was maximal in the approximate interval around $\ddot{x}_{P,n}$ = −2.5 m/s^2 (Figure 5). Different values of $\ddot{x}_{P,n}$ were predicted to be accompanied by different fluctuations, as quantified in the empirical measure $SD\ddot{x}_{P,n}$. According to Prediction 2, $SD\ddot{x}_{P,n}$ should increase with decreasing $\Sigma\Delta L$. To test the subjects' performance along this prediction, we plotted $SD\ddot{x}_{P,n}$ against $\ddot{x}_{P,n}$ for all subjects and all conditions (Figure 10). Although it was shown in Equation 6 that the limits of the stability estimates depend on g, and also that $\Sigma\Delta L$ is dependent on g, a numerical nonlocal stability analysis, performed for the two values of gravity from the experiment, produced only marginal differences. Therefore, it was permissible for the two gravity conditions to be pooled for the present evaluation. In Figure 10, the experimental data of $SD\ddot{x}_{P,n}$ are plotted together with the numerical results of $\Sigma\Delta L$. The latter data were scaled to match the range of the experimental values. Qualitatively, it is evident that the distribution of $SD\ddot{x}_{P,n}$ data follows the shape of the numerically derived curve. A quantitative assessment of the data distribution was obtained from a second-order polynomial regression, which gained a significant fit, r^2 =.32, $y = 1.43 + 0.05x + 0.02x^2$, $p <$ .0001; the squared term contributed with a probability of $p < .02$. As the global stability curve could also be fitted significantly by a second-order polynomial (r^2 = .90), the

TABLE 1
Mean Values of Paddle Amplitude, $x_{P,amp}$ (m), for 6 Subjects at Three Juggling Heights (Low, Medium, and High) and Two Gravity Conditions, G_{normal} and $G_{reduced}$

	G_{normal}				$G_{reduced}$			
Subject	Low	Medium	High	Control	Low	Medium	High	Control
1	.053	.109	.149	.300	.055	.108	.149	.295
2	.064	.092	.165	.136	.068	.095	.183	.132
3	.036	.099	.216	.190	.042	.111	.216	.122
4	.134	.158	.217	.181	.106	.141	.240	.155
5	.032	.074	.103	.205	.031	.074	.122	.192
6	.061	.093	.160	.106	.074	.113	.170	.125

Note. In the control condition, paddle movement was performed but no ball was juggled.

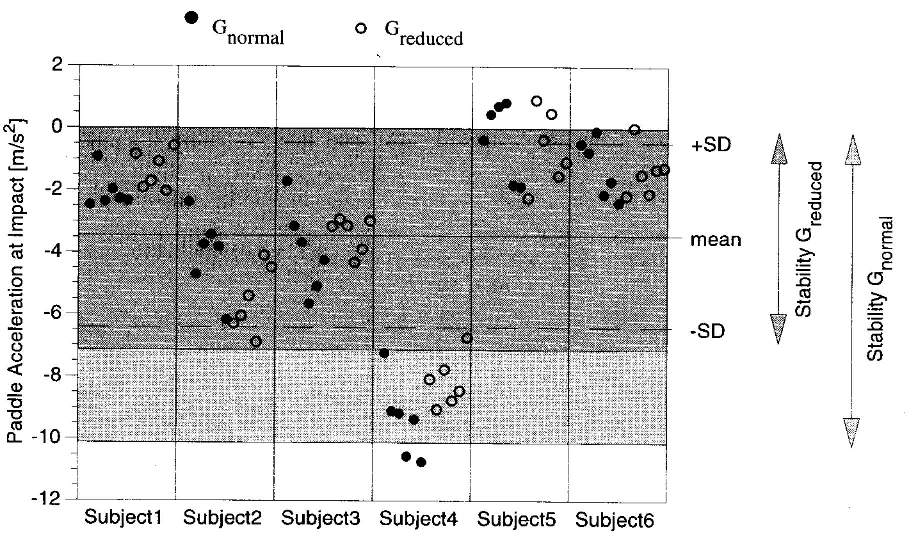

FIGURE 9. Trial means of acceleration values at impact, $\ddot{x}_{P,n}$, for all six experimental conditions, grouped by subject. The symbols differentiate the data for the two gravity conditions. The dark shading covers the range of maximal local stability for $G_{reduced}$, the light shading the range of maximal stability for G_{normal}. The overall mean and its standard deviation refer to the mean across all subjects and all conditions.

qualitative similarity of the two results was confirmed. This result supports Prediction 2: that stable juggling was indeed governed by the task's dynamics and not by an imposed control mechanism.

Dynamical Scaling of the Paddle Movement

Because for each individual subject the stability criterion, $\ddot{x}_{P,n}$, did not change significantly with the experimental manipulations, we investigated how subjects modify the movements of the paddle so as to juggle the ball at different heights. The continuous kinematic characteristics of the paddle movement were captured by its amplitude, $x_{P,amp}$, and amplitude of velocity, $\dot{x}_{P,amp}$. When the same two-way analysis of variance as above was conducted on paddle amplitude, $x_{P,amp}$, it revealed that $x_{P,amp}$ increased significantly with increasing ball amplitude, $x_{B,amp}$. Prediction 3 states that if subjects find dynamically stable solutions and hold the acceleration at impact invariant across the experimental conditions, $x_{P,amp}$ should be proportional to the square of paddle period, τ^2, and $\dot{x}_{P,amp}$ should be proportional to τ.

These predictions were tested by means of linear regression analysis. Figure 11 shows the results for the scaling between $x_{P,amp}$ and τ^2 for all 6 subjects individually. When the regression was performed separately for G_{normal} and $G_{reduced}$, all subjects obtained highly significant fits with r^2 values ranging from .92 to .99. The different slopes that each subject displayed for G_{normal} and $G_{reduced}$ in the six conditions were entered into a two-tailed t test. The result was significant, $t(5) = 3.1$, $p < .05$, meaning that the ball's flight properties led to different scaling constants, c. Similarly significant regressions were obtained when $\dot{x}_{P,amp}$ was regressed against τ. These results of the regression analyses are summarized in Table 2. The highly significant fits supported

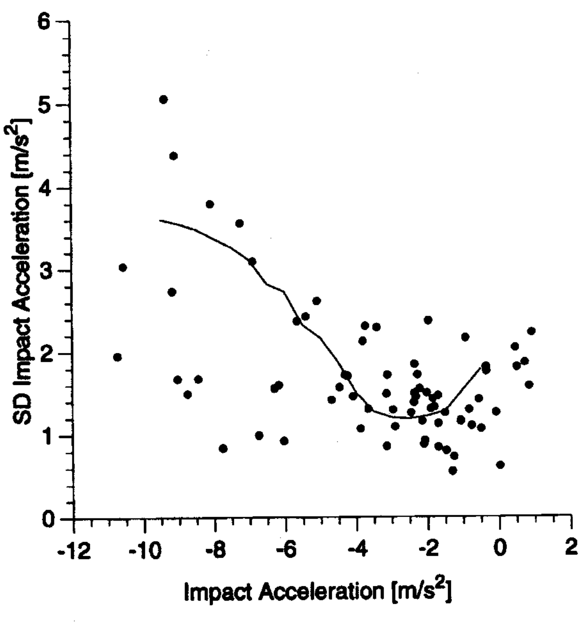

FIGURE 10. Standard deviations of 6 subjects in six experimental conditions plotted against the mean acceleration at impact, $\ddot{x}_{P,n}$. We scaled the graph of Figure 5 appropriately and inserted it to allow a qualitative comparison.

Prediction 3, indicating that the paddle trajectories of each subject for each gravity condition can be normalized to one trajectory by the scaling relation *h*.

To further see the effect of *h* on the continuous trajectories, we performed the scaling transformation on the continuous data. Figure 12 illustrates this normalization on the 12 trials of Subject 1 for G_{normal} and $G_{reduced}$, respectively. The two panels on the left (a, c) show six original mean limit cy-

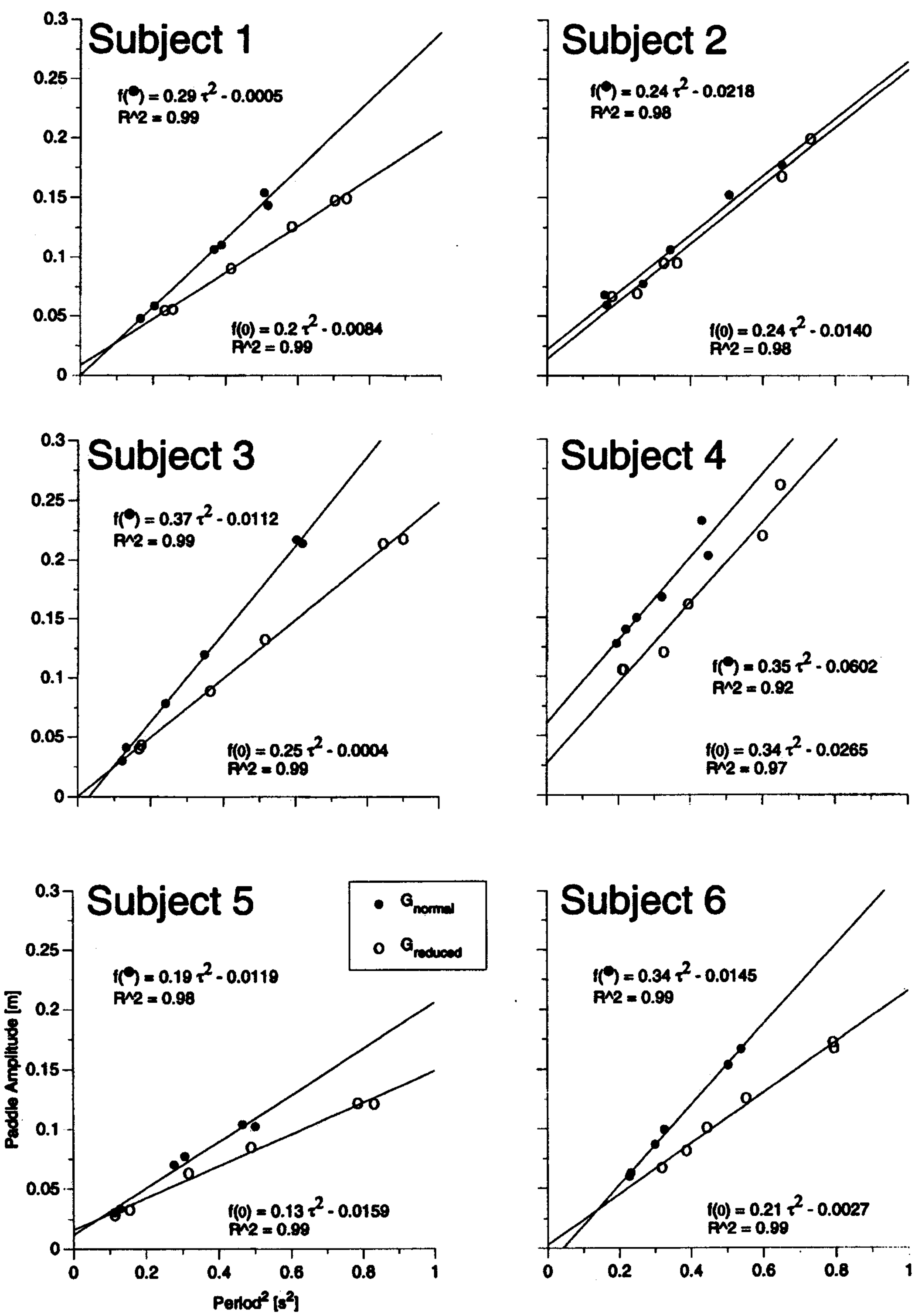

FIGURE 11. Trial means of paddle amplitude $x_{P,amp}$ were regressed against the squared period, τ^2. The data were pooled over different juggling heights, but separated for the two gravity conditions.

cles paired in three distinct bands, which corresponded to the three juggling heights; the two panels on the right (b, d) show the same limit cycles after this normalization. As can be seen, by applying h, the six limit cycles were collapsed onto a narrow band. In a first approximation, this can be understood as evidence that human subjects scale their trajectories according to the dynamical scaling described above.

Discussion

In the present study, we analyzed the skill of juggling a ball rhythmically on a racket from the perspective of dynamical systems theory. The analysis of the difference equations modeling this dynamical system demonstrated that the task dynamics offer an economical juggling pattern that is stable even for open-loop actuator motion. Our goal

TABLE 2
Regression Results of Paddle Velocity, $\dot{x}_{P,amp}$, Over Paddle Period, τ

	G_{normal}			$G_{reduced}$		
Subject	R^2	Intercept	Slope	R^2	Intercept	Slope
1	.978	–.141	2.163	.973	–.032	1.502
2	.889	.312	1.564	.925	.136	1.704
3	.994	–.511	3.088	.993	–.256	2.068
4	.589	1.084	1.54	.828	.515	1.822
5	.981	.128	1.367	.997	.143	1.047
6	.992	–.495	2.756	.983	–.257	1.831

Note. Regressions were conducted separately for the two gravity conditions, G_{normal} and $G_{reduced}$.

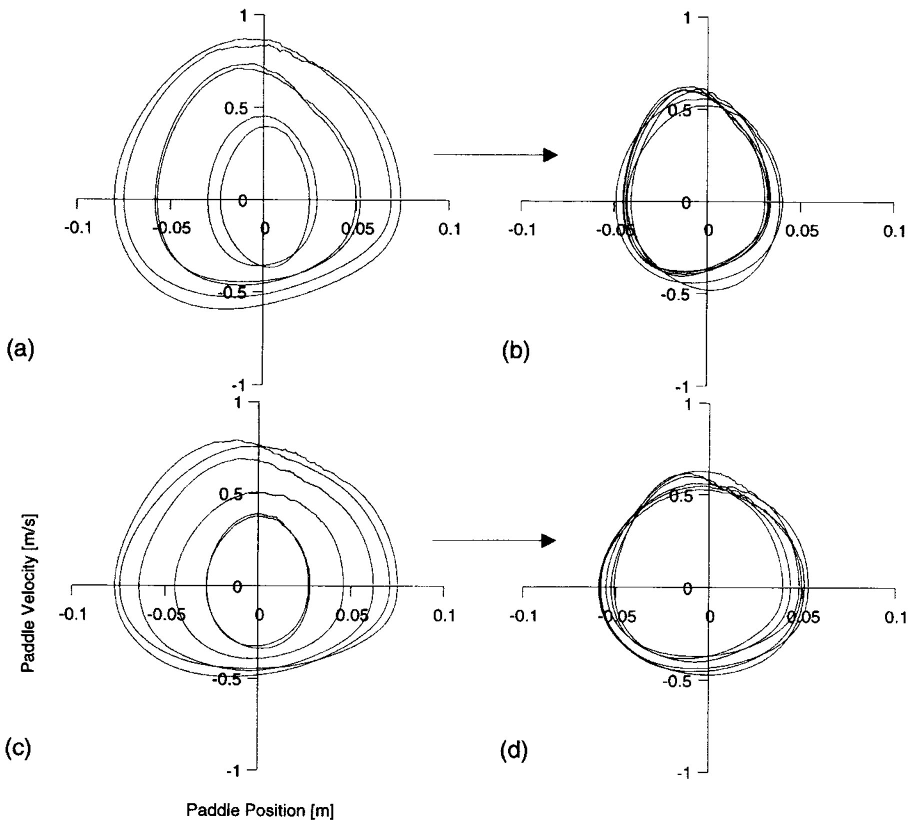

FIGURE 12. Application of the scaling relation in the phase portrait on the 12 mean trajectories of Subject 1. **(a)** Trials performed under normal gravity conditions; **(c)** trials performed under reduced gravity conditions. The unscaled plots in **(a)** and **(c)** were normalized by h and are shown in **(b)** and **(d)**, respectively.

in this study was to investigate whether principles determining this behavior are also relevant for humans performing a comparable task.

Analytical Results

The mathematical analysis of the model equations in which a ball bounces on a rhythmically moving planar surface, without perceptual feedback, yielded two major results identifying (a) the characteristic criteria for stability of this periodic motion and (b) a scaling relation that acts as an organizing principle for a class of movement variants. The interaction between ball and surface was shown to display dynamically stable fixed points that required the actuator's movement to be in its decelerating phase at the moment of impact. The range of values of deceleration that ensure this stability is bounded by analytically determined limits. Within these bounds, the different values can be assigned differential stability by means of local and nonlocal stability analysis. If the dynamical system is analyzed with respect to topological orbital equivalence (TOE), a family of solutions to the periodic juggling task are found that are essentially equivalent, because they are governed by identical dynamical properties. Trajectories with different amplitudes and periods are transformable into each other by a single scaling relation and are therefore invariant.

Empirical Results

These two fundamental results about the dynamical model system provided hypotheses for investigating how human subjects perform the task. The question guiding the experiments was to what extent human movement takes advantage of the task dynamics. Because the analysis of the model system revealed that the nonreactive, that is, open-loop, motion can achieve a stable bouncing pattern, the question was whether human control mimics this strategy or, more accurately, tunes into this passively stable pattern to obtain a parsimonious solution of the task. Importantly, the requirement for deceleration of the paddle at impact that characterizes the passively stable solution is nonobvious for an external planning device. If human coordination of hand and ball is indeed comparable with the equilibrium solution of the dynamical model system, then an actor can be viewed as an equal part of such a coupled dynamical system. This perspective is in contrast to the view that attributes to the human actor a dominant role that is logically distinct from the dynamical system, where the human actor would impose control over the system. In this framework, arbitrarily many different solutions can be generated by making use of feedback stabilization. Such solutions are qualitatively different from the passively stable solutions brought about by coupled dynamical systems, in which the actor is seen as participating in and coupled to the entire action system.

The results of the experiment in which subjects performed one-handed paddle juggling can be summarized as follows:

1. Subjects juggled the ball with negative accelerations of the paddle at the moment of impact, $\ddot{x}_{P,n} < 0$.
2. Different task manipulations, three juggling amplitudes, and two gravitational conditions for the ball did not influence the paddle acceleration at impact.
3. Subjects chose individually different negative impact accelerations of the paddle as their preferred juggling regime.
4. The standard deviation of the paddle's acceleration at impact, $SD\ddot{x}_{P,n}$, varied as a function of the magnitude of the paddle's acceleration, in accordance with results from the numerical nonlocal stability analysis.
5. The constant ratios of paddle position amplitude against paddle period and paddle velocity amplitude against paddle period suggest that the subjects' trajectories abide by a dynamical scaling rule when the ball is juggled at different heights.

Results 1 to 4 support the conclusion that humans indeed find a solution to the task in the inherent dynamics that does not require additional corrective control mechanisms by means of continuous perceptual coupling. Result 5 opens up a another perspective on the role of dynamical principles. Each of these will be dealt with in turn.

Motor Equivalence

The scaling relation h first was a necessary fact to facilitate our analysis of different realizations of juggling patterns but, because of Result 5, it also allows a more far-reaching interpretation. As described earlier, the scaling relation is a nonlinear transformation that is dependent on only one parameter. It can serve to generate arbitrarily many realizations that are functionally equivalent by a low-dimensional relation, or conversely, it transforms many different spatiotemporal realizations into one structurally equivalent fundamental movement pattern. This one-to-many mapping is the essence of Bernstein's (1967) principle of motor equivalence, which says that when a variety of patterns that are seemingly different on the level of kinematic description are functionally equivalent, then their organizational principle should be the same. Within the dynamical perspective to movement control this intuition has been interpreted as the requirement of one unifying control principle, which should be brought about by nonspecific parameter changes (Turvey, 1977) and which should be low dimensional (Haken & Wunderlin, 1990). In the present work, the scaling relation is an explicit formulation of this notion of functional equivalence, and its validity was empirically verified with Result 5. The underlying theorem of TOE can therefore be seen as the dynamical rationale for the organizing principle of motor equivalence, in the sense of Bernstein.

Some aspects of the data on the scaling relation, however, require further clarification. It was shown that the paddle amplitudes, $x_{P,amp}$, for the three juggling heights significantly scale to the squared paddle period τ^2 (Figure 11). Still, it re-

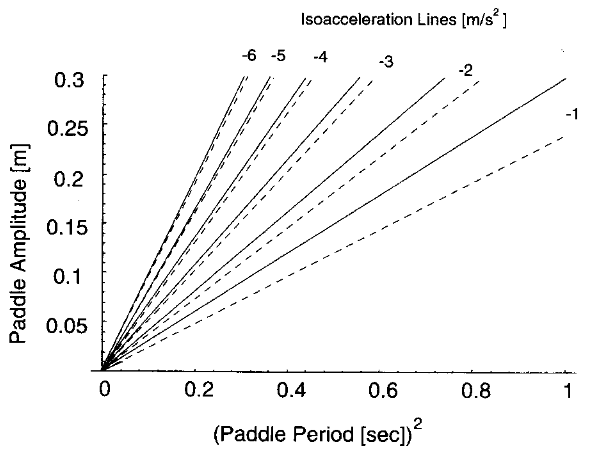

FIGURE 13. Mean paddle amplitude as function of squared mean paddle period for a sinusoidal driving motion of the paddle if the paddle acceleration at impact is kept constant over different juggling heights. The solid lines represent the normal gravity condition; the dashed lines, the reduced gravity condition.

mains to be explained why the slopes of the regression lines were different in value for the 6 subjects and why the two gravity conditions seemed to produce different scaling constants. Whereas for all subjects the slopes for $G_{reduced}$ showed a trend to be smaller than G_{normal}, the two regression lines diverged for 4 subjects, but were parallel for 2 subjects (Subjects 2 and 4). That these seemingly different results do not contradict the main conclusion about the role of the scaling relation can be explained with an example. For an idealized sinusoidal paddle movement, amplitudes and periods for different accelerations $\ddot{x}_{P,n}$ were generated and their scaling constant was determined (see Appendix). The results of this calculation are presented in Figure 13, which shows six pairs of lines for six different $\ddot{x}_{P,n}$ at G_{normal} and $G_{reduced}$, respectively. Each line was the result for one choice of $\ddot{x}_{P,n}$ and one gravity condition when the paddle movement was scaled according to relation h. It can be seen that (a) the value of the scaling constant is a function of $\ddot{x}_{P,n}$ and that (b) the slopes were smaller for $G_{reduced}$. Interestingly, for higher $\ddot{x}_{P,n}$, the difference between the two gravity conditions became increasingly smaller, rendering the lines almost parallel. Based on the result that each subject had one characteristic $\ddot{x}_{P,n}$, and that they differed from each other, this numerical study can explain the apparent interindividual differences: The regression slopes of Subject 2 and Subject 4 were almost parallel, because both subjects juggled at a relatively high $\ddot{x}_{P,n}$ (Subject 2: –4.8 m/s^2; Subject 4: –8.75 m/s^2). All other subjects' mean $\ddot{x}_{P,n}$ remained below –3.0 m/s^2, where the slope differences were shown to be greater. Thus, these considerations, which explicate the quantitative intra- and intersubject differences from Figure 11 even further, support the invariance postulated by the scaling relation in accordance with the principle of motor equivalence.

Reach of the Dynamical Strategy

Despite the fact that, on the whole, the experimental results supported the hypothesis of a dynamical system's solution, we still need to consider whether the data are sufficient to exclude alternative interpretations and, of course, whether alternative solutions exist. One alternative strategy may be that subjects exploit continuous closed-loop control with feedback. In the present case, only the moment of ball–paddle contact permitted corrective control of the ball's trajectory. For a biological system, however, this feedback control is not feasible, because the duration of an impact lies in the range of 10–20 ms, which is too short a time even for spinal reflexes. Therefore, the impact itself cannot be regulated. However, feedback regulation may become possible when anticipatory control is applied. This type of control was implemented in the robotics study by Aboaf et al. (1989), where, on the basis of the observed ballistic flight of the ball, the next impact of ball and paddle was planned ahead. Similar strategies might be exploited by human subjects. For instance, Figure 14 depicts one of the five trials in which Subject 5 juggled the ball with positive paddle acceleration at the point of impact—contrary to the model predictions. On the basis of the given dynamical analysis, such a solution should be unstable and the pattern should be lost. As this was not the case, some additional control mechanisms have to be assumed. As laid out above, the model system did not include a coupling from ball to paddle and the remaining control may be found in the perceptual component. The nature of such perceptual coupling between ball and paddle is being addressed in further research. Importantly, though, these trials prove that the sub-

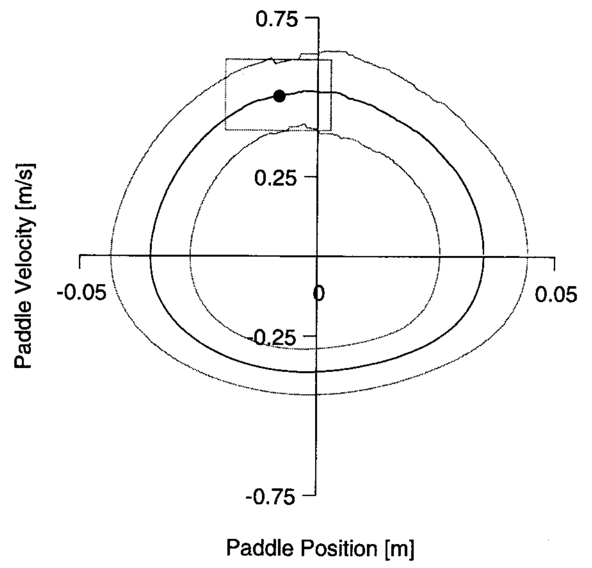

FIGURE 14. A juggling trial in which the task goal was achieved with positive paddle accelerations at impact($\ddot{x}_{P,n}$ = +2 m/s^2).

jects' solutions characterized by negative acceleration at impact are not the only solution and the fact that all other subjects reliably juggled with negative acceleration at impact confirms that there is a common strategy that is captured by the given dynamical model.

Conclusions

The results of the experiment demonstrate that biological systems can exploit the inherent dynamics of a task by using the dynamically stable solutions defined by the task. Hence, successful performance of a movement skill does not necessarily require extensive real-time planning of trajectories and computationally expensive information processing. From the viewpoint of perception–action coupling, these results state that a coupling mechanism between perception and action can be brought about in which the inherent dynamics of a task are not destroyed if they support the task goal. The data also support the conclusion that human subjects scale their movements according to a scaling relation that leaves dynamical properties invariant, despite individually different spatiotemporal realizations. This scaling can be considered as an explicit formulation for the principle of motor equivalence, a higher order invariance by which the system generates variants of a skill providing for the biologically requisite flexibility.

In the present analysis of the discretized model equations, it was implicit that ball and actuator form a nonlinear system of coupled oscillators. We will pursue this issue further in future work, by analyzing the continuous spatiotemporal structure of the component units and the coupled system, to shed light on the nature and the role of perception–action coupling in this task.

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APPENDIX

Assuming an idealized sinusoidal juggling movement, it can be demonstrated that under the scaling relation h the regression lines of Figure 11 should be straight where the slopes are a function of $\ddot{x}_{P,n}$, g, and α:

$$\frac{x_{P,amp}}{\tau^2} = f(\ddot{x}_{P,n}, g, \alpha) = constant. \tag{A1}$$

This will be derived as follows. For a sinusoidal paddle movement, the paddle velocity is given by

$$\dot{x}_P = \frac{x_{P,amp}}{2}\omega\cos(\omega t), \qquad \omega = \frac{2\pi}{\tau},$$

which can be reformulated as

$$\frac{x_{P,amp}}{\tau^2} = \frac{\dot{x}_P}{\pi\tau\cos(\omega t)}. \tag{A2}$$

For stable juggling, Equation (4[ii]) gives an expression for $\dot{x}_{P,n}$, which can be inserted into Equation (A2) as follows:

$$\frac{x_{P,amp}}{\tau^2} = \frac{0.5g\tau\frac{1-\alpha}{1+\alpha}}{\pi\tau\cos(\omega t_{impact})} = \frac{g(1-\alpha)}{2\pi(1+\alpha)\cos(\omega t_{impact})}. \tag{A3}$$

As the last step, the cosine term in Equation (A3) must be replaced by a more general expression. Here it is useful to start with the quotient of paddle acceleration and paddle velocity:

$$\frac{\ddot{x}_{P,n}}{\dot{x}_{P,n}} = \frac{-x_{P,amp}\omega^2\sin(\omega t_{impact})}{x_{P,amp}\omega\cos(\omega t_{impact})} = -\omega\tan(\omega t_{impact}),$$

and to insert Equation (4[ii]) in this expression:

$$\tan(\omega t_{impact}) = -\frac{\ddot{x}_{P,n}}{\omega\dot{x}_{P,n}} = -\frac{\ddot{x}_{P,n}(1+\alpha)}{\omega\dot{x}_{B,n}(1-\alpha)}. \tag{A4}$$

For stable juggling, the impact velocity of the ball, $\dot{x}_{B,n}$, is entirely determined by the ballistic flight to be $\dot{x}_{B,n} = -0.5g\tau$ Combining this result with Equation (A4) gives

$$\tan(\omega t_{impact}) = \frac{\ddot{x}_{P,n}(1+\alpha)}{\pi g(1-\alpha)}. \tag{A5}$$

Inserting Equation (A3) into Equation (A5) results in the final expression:

$$\frac{x_{P,amp}}{\tau^2} = \frac{g(1-\alpha)}{4\pi(1+\alpha)\cos\left(\arctan\left(\frac{\ddot{x}_{P,n}(1+\alpha)}{\pi g(1-\alpha)}\right)\right)} = constant. \tag{A6}$$

Equation (A6) depends only on the constants g and α, and the paddle acceleration at impact, $\ddot{x}_{P,n}$. Because $\ddot{x}_{P,n}$ remains constant under the scaling relation h, Equation (A6) must be constant across different realizations of juggling. This demonstrates that for sinusoidal juggling trajectories, plotting $x_{P,amp}$ against τ^2 should yield straight lines. Varying g according to the experimental conditions results in the plots of Figure 13. This result generalizes to arbitrary periodic trajectories.

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