16-299 Introduction to Feedback Control Systems
Spring 2021
Midterm Exam

Instructions:
Open note, open book, open internet.
You may use any resource except for other humans.
Please show all work.
Intermediate steps must be legible and answers must be explained to receive credit.
Your solution must be submitted as a pdf file to Canvas by
2:20pm EDT on Wednesday March 17

1. (15 points) Write the scalar ODE below in state space form.

$$\dddot{y} + \gamma \ddot{y} - (\dot{y} - y^2)^2 = u.$$

2. (15 points) Derive the conditions under which the observer state estimate $\hat{x}(t)$ converges to the actual state $x(t)$, where $\hat{x}(t)$ is found by integrating the equation

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K_o(C\hat{x}(t) - y(t)),$$

and $x(t)$, $u(t)$, and $y(t)$ are the state, input, and output to the usual continuous time state space system:

$$\dot{x}(t) = Ax(t) + Bu(t),$$

$$y(t) = Cx(t).$$

Note that this problem is slightly trickier than it looks, you cannot just copy the answer from the lecture notes since the definition of the observer is slightly different. You will need to re-derive it on your own.

3. Consider the nonlinear system

$$\dot{x} = \begin{bmatrix} x_2^2 + x_1 \cos x_2 - u \\ x_2 + (x_1 + 1)x_1 + x_1 \sin x_2 + u \end{bmatrix}$$

   (a) (5 points) Verify that there is an equilibrium point at $x = 0$.

   (b) (15 points) Linearize the system about the equilibrium point at the origin.

   (c) (5 points) What can you say about the stability of the linearized system for the equilibrium point at 0? (i.e., is it stable? unstable? asymptotically stable? can't tell?)

4. (5 points) Is the equilibrium point at zero stable for the unforced state space system below stable? Explain your answer.

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -3 & -6 \end{bmatrix} x$$

5. (10 points) Is the system below controllable? Explain your answer.

$$\dot{x} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

6. (15 points) Consider the linear state space system

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -3 & -6 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 4 & 5 & 6 \\ 0 & 0 & 7 & 8 & 9 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u,$$

where the full state can be measured. Use eigenvalue placement to design a feedback control law $u = -Kx$ so that the resulting closed loop system is asymptotically stable. (hint: this is trickier than it looks, as you will quickly discover if you try to blindly use the MATLAB place command. You may find some solace in the block diagonal structure of the A matrix, the answers to the previous two problems, and the general advice to let go of the things you cannot control.)

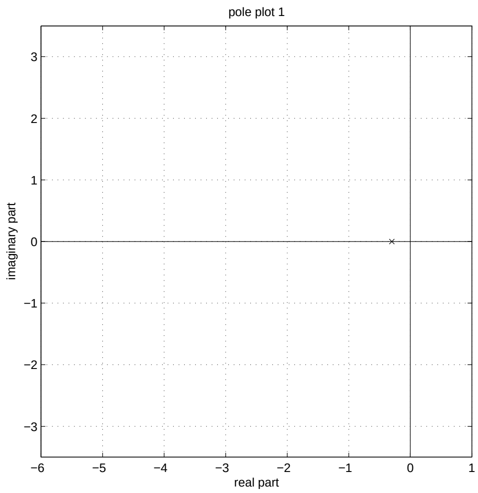
7. (20 points) The following two pages contain eigenvalue location plots and step response plots for 4 different systems. Your job is to figure out which plot goes with which. Each answer is worth 5 points: 1 point for choosing the correct plot and 4 points for giving the correct explanation.

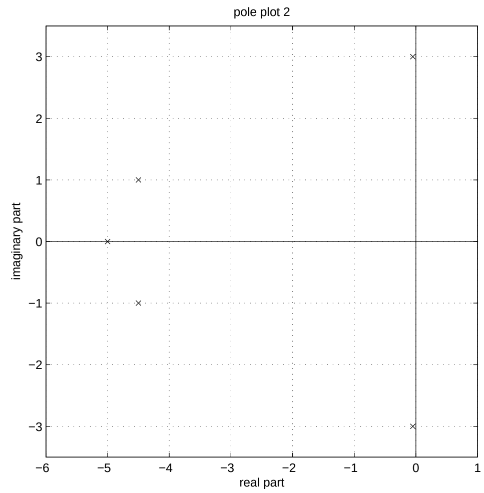
   (a) Which step response goes with eigenvalue plot 1? Why?

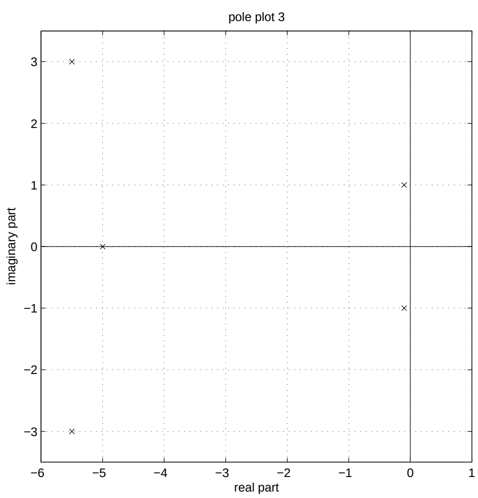
   (b) Which step response goes with eigenvalue plot 2? Why?

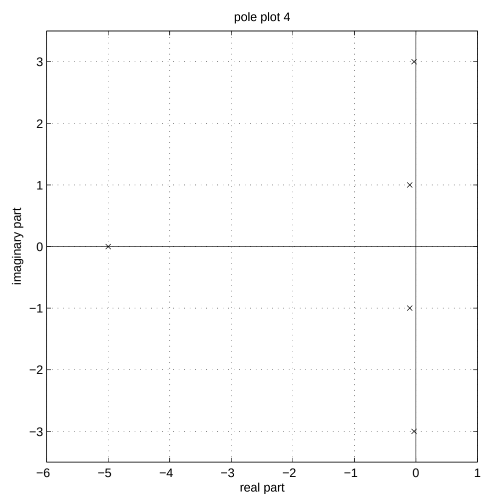
   (c) Which step response goes with eigenvalue plot 3? Why?

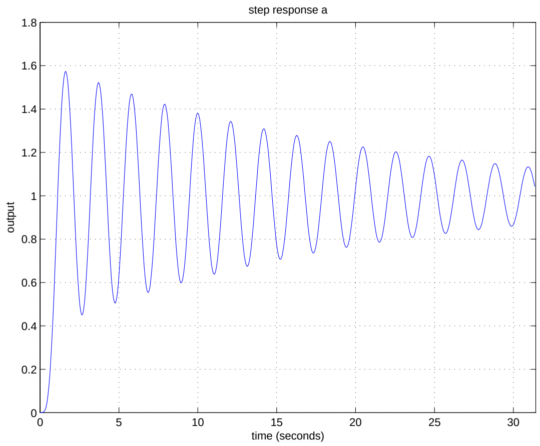
   (d) Which step response goes with eigenvalue plot 4? Why?

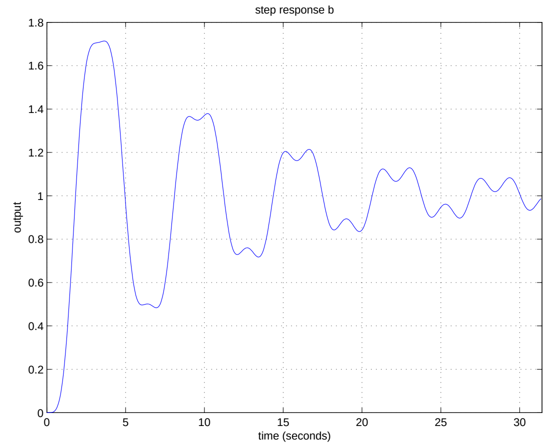

pole plot 1
imaginary part
real part

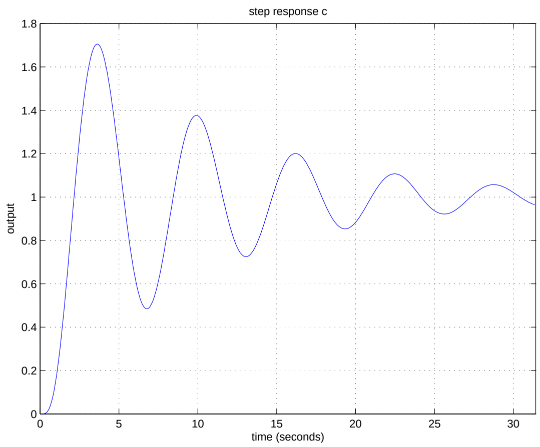

pole plot 2
imaginary part
real part

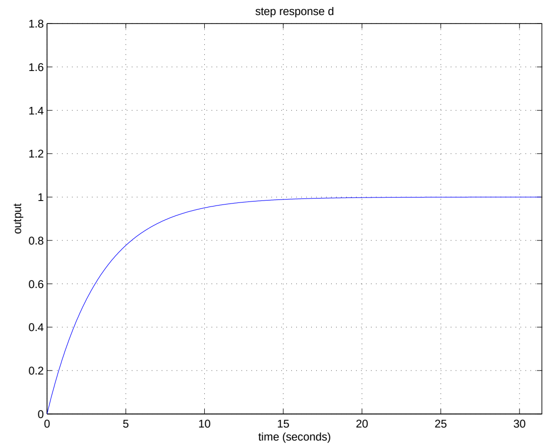

pole plot 3
imaginary part
real part


pole plot 4
imaginary part
real part


step response a
output
time (seconds)
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
0
5
10
15
20
25
30


step response b
output
time (seconds)
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
0
5
10
15
20
25
30


step response c
output
time (seconds)
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
0
5
10
15
20
25
30


step response d
output
time (seconds)
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
0
5
10
15
20
25
30