

# 16-299: Introduction to Feedback Control Systems

Spring 2021

## Problem Set 3

due 8 April 2021

Please show all work and write clearly—all intermediate steps must be shown and legible. Submit solutions as a zip file that contains written explanations of problem solutions and relevant MATLAB code.

You are encouraged to work with other students to find the answers to these problems, however each student must submit his/her own *unique* set of solutions. If you do work with other students, please give them credit by listing their names at the end of your solutions.

In the system shown in Fig. 1, a bar is free to rotate about a pivot, with a mass and a propeller attached at the two ends of the bar. The propeller is driven by an attached motor. The input  $u$  to the system is the voltage applied to the propeller motor and the outputs  $\theta$  and  $\omega$  are the angle of the bar and the angular speed of the propeller respectively. The center of mass  $m$  of the system is at a distance  $l_m$  from the pivot. The motor-driven propeller provides downward thrust (exerted on the bar) to counter the weight at the other end of the bar. In this assignment, you will experience the joy of writing a complete non-linear simulator for this system.

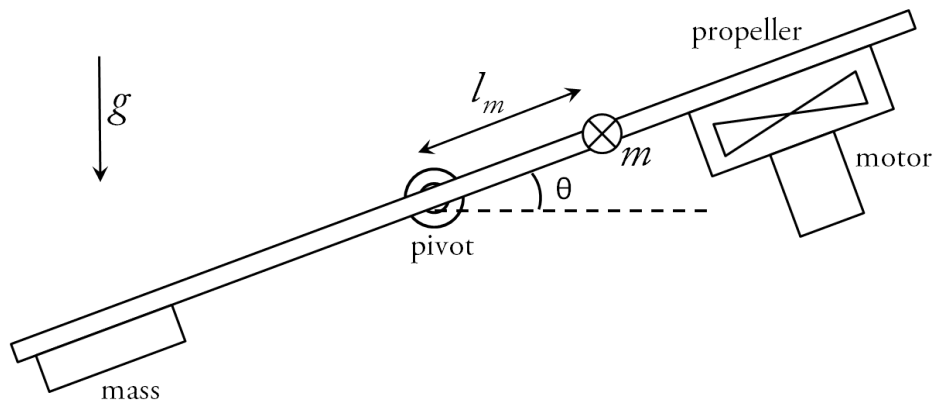


Figure 1: Mass - Propeller System.

The system dynamics is given by the following differential equations:

$$J \frac{d^2}{dt^2} \theta(t) + \beta \frac{d}{dt} \theta(t) = -l_f k_\omega \omega(t) - m l_m g \cos(\theta(t)) \quad (1)$$

$$u(t) = \frac{R}{k_i} \left( J_f \frac{d}{dt} \omega(t) + k_d \omega(t) \right) + k_v \omega(t) \quad (2)$$

where,

- $u(t)$  is the input motor voltage (in volts),
- $\omega(t)$  is the propeller speed (in rad/s),
- $\theta(t)$  is the angle the mass-propeller system makes with reference to the horizontal (in radians),
- $J = 0.0015\text{kg}\cdot\text{m}^2$  is the moment of inertia of the mass-propeller system about the pivot point,
- $\beta = 0.01\text{N}\cdot\text{m}/\text{rad}/\text{s}$  is the rotational coefficient of damping of the mass-propeller system about the pivot point,
- $m = 0.15\text{kg}$  is the total mass of the system,
- $l_m = -0.05\text{m}$  is the location of the center of mass of the system relative to the pivot point (-ve value indicates it is to the left of the pivot),
- $l_f = 0.1\text{m}$  is the location of the center of thrust of the propeller relative to the pivot point,
- $k_\omega = 6.75 \times 10^{-3}\text{N}/\text{rad}/\text{s}$  is the propeller thrust to propeller angular speed ratio,
- $R = 1\Omega$  is the motor coil resistance,
- $J_f = 2 \times 10^{-6}\text{kg}\cdot\text{m}^2$  is the moment of inertia of the motor and propeller about the motor axis,
- $k_i = 7 \times 10^{-3}\text{N}\cdot\text{m}/\text{A}$  is the motor current-torque constant,
- $k_v = 0.06\text{V}/\text{rad}/\text{s}$  is the motor back-emf constant,
- $k_d = 2 \times 10^{-6}\text{s}^{-1}$  is the propeller air drag constant,
- $g = 9.8\text{m}/\text{s}^2$  is the acceleration due to gravity.

Use this model to complete each of the following problems:

1. (5 points) Define a state vector  $x$  and rewrite the two ODEs as a first order vector ODE of the form  $\dot{x} = f(x, u)$ .
2. (15 points) Write a MATLAB program that uses fourth order Runge-Kutta method to numerically solves for  $\theta(t)$  and  $\omega(t)$ . Assume that the initial condition is  $\dot{\omega}(0) = \omega(0) = 0$ , and  $\dot{\theta}(0) = \theta(0) = \omega(0) = 0$ . The input is a step function, i.e.,

$$u(t) = \begin{cases} 0 & \text{if } t < 0 \\ 6.07 & \text{if } t \geq 0 \end{cases}$$

Use a time step of  $T = 0.01$ , and integrate the equations from  $t = 0$  to  $t = 5$  seconds. Repeat the simulation, with no input ( $u(t) = 0\forall t$ ). Attach a hardcopy of your code and plots of  $\theta$  and  $\omega$  v.s. time.

3. (10 points) For a sinusoidal input  $u(t) = 6.07 + 0.5\sin(2\pi ft)$  with  $f = 0.1$  Hz, integrate the differential equations for 40 seconds (use  $T = 0.01$ ). Plot the input signal  $u(t)$ ,  $\omega(t)$  and  $\theta$  as functions of time. Compare the amplitude of  $u(t)$  with the amplitude of  $\theta(t)$  and estimate the gain  $A_{0.1}$  and phase shift  $\phi_{0.1}$  (in degrees) of the system from input voltage to  $\theta$ .
4. (10 points) Repeat Problem 3 to find  $A_f$  and  $\phi_f$  for  $f$  equal to 0.01, 0.03, 0.08, 0.1, 0.3, 0.8 and 1.0 Hz. You don't need to show the plots, just compute the phase shift and gain for each frequency. (You will need to integrate for longer than 40 seconds for  $f = 0.05$  and 0.08.)
5. (10 points) Using the results of Problems 3 and 4, generate a plot of  $20\log_{10}(A_f)$  vs.  $f$ , where  $f$  should be plotted on a logarithmic scale ( $20\log_{10}(A_f)$  is the gain represented in units of *decibels*.) *Some hints: you can use the MATLAB functions `logspace` and `semilogx` to generate nice looking plots.*

6. (10 points) Using the results of Problems 3 and 4, generate a plot of  $\phi_f$  vs.  $f$ , with  $f$  plotted on a logarithmic scale.
7. (10 points) Find the transfer functions  $H_1(s) = \frac{\Theta(s)}{\Omega(s)}$ ,  $H_2(s) = \frac{\Omega(s)}{U(s)}$  and hence  $H(s) = \frac{\Theta(s)}{U(s)}$ . Here,  $\Theta(s) = \mathcal{L}_t[\theta(t)](s)$  and  $\Omega(s) = \mathcal{L}_t[\omega(t)](s)$  are the Laplace transforms of  $\theta$  and  $\omega$  respectively. Use the following approximations to Eqs. 1,2 :

$$J \frac{d^2}{dt^2} \theta(t) + \beta \frac{d}{dt} \theta(t) = -l_f k_\omega \omega(t) \quad (3)$$

$$u(t) = \frac{R}{k_i} \left( J_f \frac{d}{dt} \omega(t) + k_d \omega(t) \right) + k_v \omega(t) \quad (4)$$

**Note:** The approximations required to derive Eqs. 3,4 from Eqs. 1,2 are:

- (a)  $\theta \rightarrow 0$  hence, correct to second order,  $\cos(\theta) \simeq 1$
- (b)  $u$  and  $\omega$  are split into  $u + u_0$  and  $\omega + \omega_0$ , where  $u_0$  is the voltage required to drive the propeller at the speed  $\omega_0$ , which just balances the mass about  $\theta = 0$ .
8. (20 points) Write a MATLAB program to generate a Bode plot of a system given its transfer function  $H(s)$ . The Bode plot is actually two plots, and those two plots are the same as the plots described in Problems 5 and 6. (i.e. the first is a plot of the gain in decibels verses frequency, the second is a plot of the phase shift verses frequency, and both plots are drawn with frequency on a logarithmic scale.) Attach a hard copy of your code. Use your program to generate the Bode plot for the system with the given parameters for frequency ranging from 0.01 to 10 Hz, and compare the result to the plots you found in Problems 5 and 6. What discrepancies do you notice? Can you explain them? (I know that MATLAB already has a Bode function in its control toolbox, if you want to use it to check your result that is fine. But I want you to write your own Bode plot function for this problem!)
9. (10 points) It is desired for a closed loop system to have a step response with a 1% settling time  $t_s$  of 2 seconds and a percent overshoot  $M_p$  of 5%. Sketch the region of the complex plane where the closed loop poles can be placed in order to meet these requirements. Assume that the closed loop system has two dominant poles.