Frequency Domain Analysis (3)

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3/1/2022

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Recap

- Laplace transform

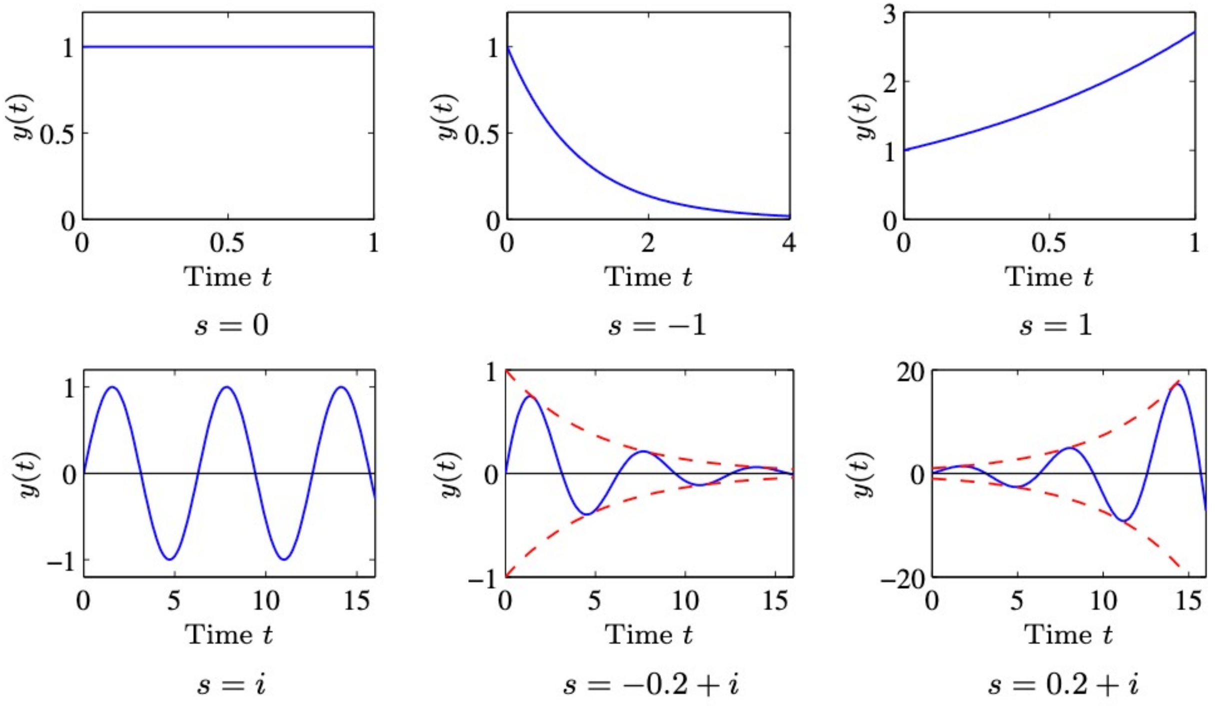
$$F(s) = \mathcal{L}\left[f(t)\right] = \int_{0_-}^{\infty} f(t)e^{-st}dt \qquad f(t) = \mathcal{L}^{-1}\left[F(s)\right] = \frac{1}{2\pi j}\int e^{st}F(s)ds$$

How to understand Laplace transform?

What's $\mathcal{L}\left[\frac{d}{dt}f(t)\right]$? $\quad sF(s) - f(0)$

What's $\mathcal{L}\left[\frac{d^n}{dt^n}f(t)\right]$?

$$s^nF(s) - \sum_{k=1}^{n} s^{n-k} \left.\frac{d^{k-1}}{dt^{k-1}}f(t)\right|_{t=0_-}$$



Recap

- Transfer function $Y(s) = \underline{H(s)}U(s)$

$H(i\omega) \longrightarrow$

$r = \|H(i\omega)\|$ gain

$\phi = \angle H(i\omega)$ Phase shift

$$y(t) = \sum_k \|H(i\omega_k)\| A_k \sin(\omega_k t + \phi_k + \angle H(i\omega_k))$$

Recap – Zeros and poles

Assuming $H(s) = \frac{b(s)}{a(s)}$ The roots of the polynomial a(s) is called the poles of the system, and roots of b(s) are called zeros of the system

Zeros: the output signal corresponding to s is zero

Poles: solution of the system when u=0 -> a mode of the system

- A system defined by a transfer function H(s) is stable if and only if all of the poles of H(s) have negative real part. Such system are said to be *Hurwitz*

Question:

Known $\mathcal{L}[f(t)] = F(s)$, what is $\mathcal{L}^{-1}F(s-a)$?

$e^{at}f(t)$

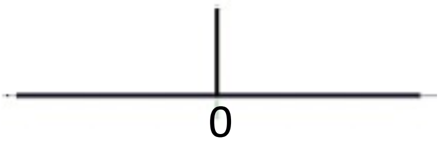
Question:

Known $\mathcal{L}[f(t)] = F(s), \mathcal{L}[g(t)] = G(s),$ what is $\mathcal{L}^{-1}[F(s)G(s)]$?

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$$

Stability

- What is stability?
- Thinking of the impulse response
- There are only two possible responses



When $t \to \infty$ $\begin{cases} y(t) \to 0 & \text{The system is } \textit{asymptotically stable} \\ y(t) \not\to 0 & \text{The system is } \textit{unstable} \end{cases}$

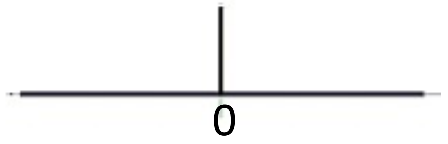
The system is *asymptotically stable*

The system is *unstable*

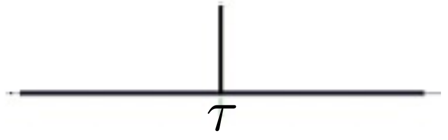
Why?

y(t) will explode since it's the sum of infinite response to impulse input

Impulse response to any response



$$\int_{-\epsilon}^{\epsilon} v(t)\delta(t) = v(0)$$



For an arbitrary f(t), there is $f(t) = \int_0^\infty f(\tau)\delta(t-\tau)d\tau$

Assuming a system has a response $\delta(t) \rightarrow y_\delta(t)$

What’s its response to f(t)?

$$\delta(t-\tau) \rightarrow y_\delta(t-\tau)$$

Because of the linear nature of the function, there will be

$$f(t) = \int_0^\infty f(\tau)\delta(t-\tau)d\tau \rightarrow \int_0^\infty f(\tau)y_\delta(t-\tau)d\tau = (f * y_\delta)(t) \longrightarrow F(s)Y_\delta(s)$$

Note that F(s) corresponds to the input of the system. What’s the transfer function?

$$H(s) = \frac{F(s)Y_\delta(s)}{F(s)} = Y_\delta(s)$$

A system’s response to an arbitrary input can be considered the weighted integral of its impulse response

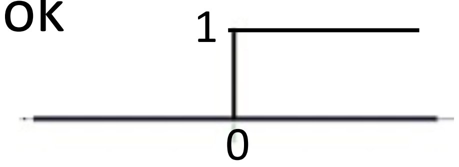
The Laplace transform of a system’s impulse response is its transfer function

Step response

- The impulse response is a simple way of characterizing the transient properties of an LTI system
  - Any input can be considered as a sum of a sequence of impulse signals. For LTI systems, the output will be the sum of the impulse response
  - Unfortunately, it is difficult to experimentally generate an impulse signal (infinitely high, infinitely narrow)

Why?

- A more commonly used method of characterizing and LTI system is to look at its *step response*



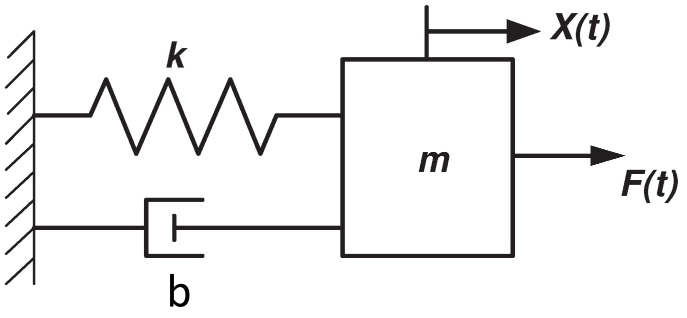
What's its Laplace form? $\frac{1}{s}$

$$s \to \infty, U(s) \to 0$$

$$u(t) = \begin{cases} 0 & \text{if} \quad t < 0 \\ 1 & \text{if} \quad t \geq 0 \end{cases}$$

Response for second order systems

- Thinking about the mass-spring-damper system



$$m\frac{d^2x(t)}{dt^2} + b\frac{dx(t)}{dt} + kx(t) = f(t)$$

$$X(s) = \frac{1}{ms^2 + bs + k}F(s)$$

What's the system's response like?

Typically written in the form of

$$H(s) = \frac{1}{ms^2 + bs + k} = \frac{K}{(s - p_1)(s - p_2)} = K\frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

Poles!

What are the poles' values?

$$p_1, p_2 = \frac{-2\xi\omega_n \pm \sqrt{4\xi^2\omega_n^2 - 4\omega_n^2}}{2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

Response for second order systems

$$H(s) = \frac{K}{(s-p_1)(s-p_2)} = K\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} \qquad p_1, p_2 = -\xi\omega_n \pm \omega_n\sqrt{\xi^2-1}$$

- In order to make the system stable, there must be Re(p_1)<0, Re(p_2)<0
- H(s) is known, what is h(t)?
  - Impulse response of the system

$$H(s) = K\frac{1}{(s-p_1)}\frac{1}{(s-p_2)}$$

Known that $\mathcal{L}(1(t)) = \frac{1}{s}$ What is $\mathcal{L}^{-1}[\frac{1}{s-p_1}]$? $e^{at}1(t)$

Convolution in temporal domain!

So, there is $h(t) = K\mathcal{L}^{-1}[\frac{1}{(s-p_1)}] * \mathcal{L}^{-1}[\frac{1}{(s-p_2)}] = K[e^{p_1 t}1(t)] * [e^{p_2 t}1(t)]$

$$= K\int_{-\infty}^{\infty}[e^{p_1\tau}1(\tau)][e^{p_2(t-\tau)}1(t-\tau)]d\tau = K\int_0^t e^{p_1\tau}e^{p_2(t-\tau)}d\tau$$

$$= K[\frac{1}{p_1-p_2}e^{p_1 t} + \frac{1}{p_2-p_1}e^{p_2 t}]$$

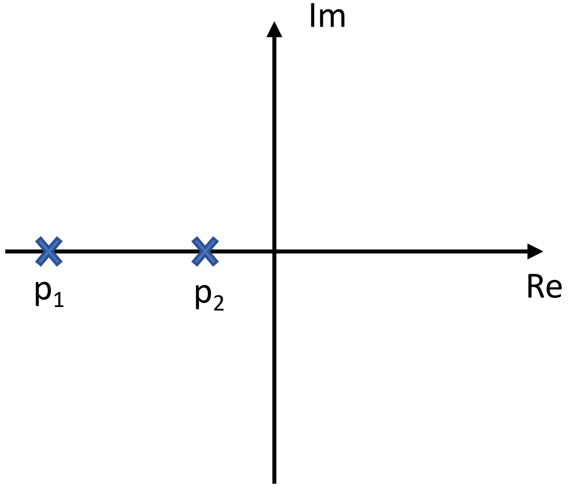
How can we understand this?

Response for second order systems

$$H(s) = \frac{K}{(s-p_1)(s-p_2)} = K\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

$$p_1, p_2 = -\xi\omega_n \pm \omega_n\sqrt{\xi^2-1}$$

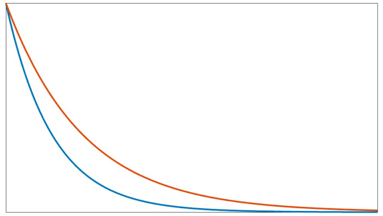
$$h(t) = K[\frac{1}{p_1-p_2}e^{p_1 t} + \frac{1}{p_2-p_1}e^{p_2 t}]$$



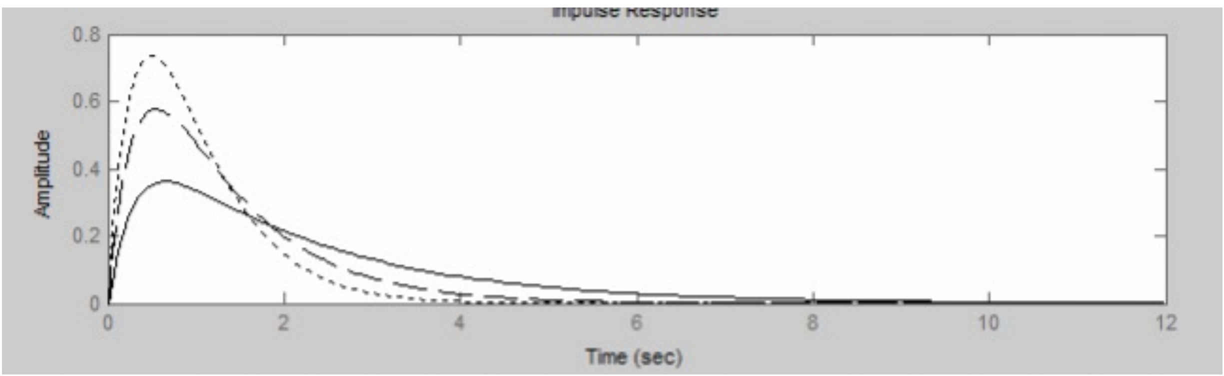
Let's consider a simplest case, p_1 and p_2 are real numbers

What's the condition? $|\xi| > 1$

What's the shape of $e^{p_1 t}, e^{p_2 t}$?



What's the shape of h(t)?

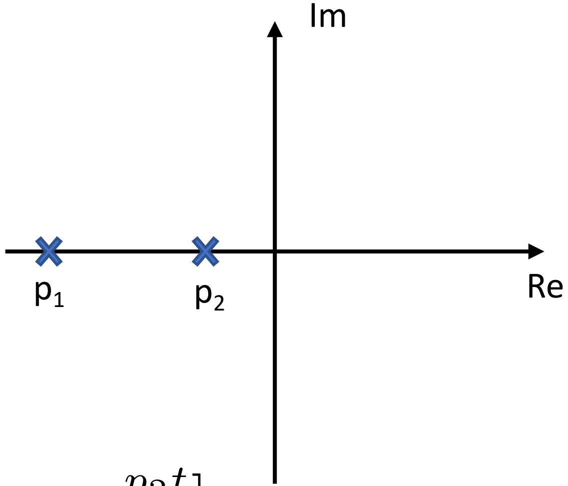


Impulse response

Response for second order systems

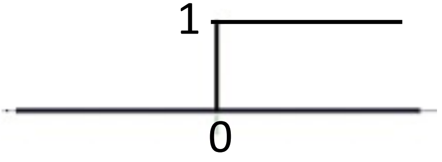
$$H(s) = \frac{K}{(s-p_1)(s-p_2)} = K\frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$p_1, p_2 = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$



$$h(t) = K[\frac{1}{p_1 - p_2}e^{p_1 t} + \frac{1}{p_2 - p_1}e^{p_2 t}]$$

How to get the step response?



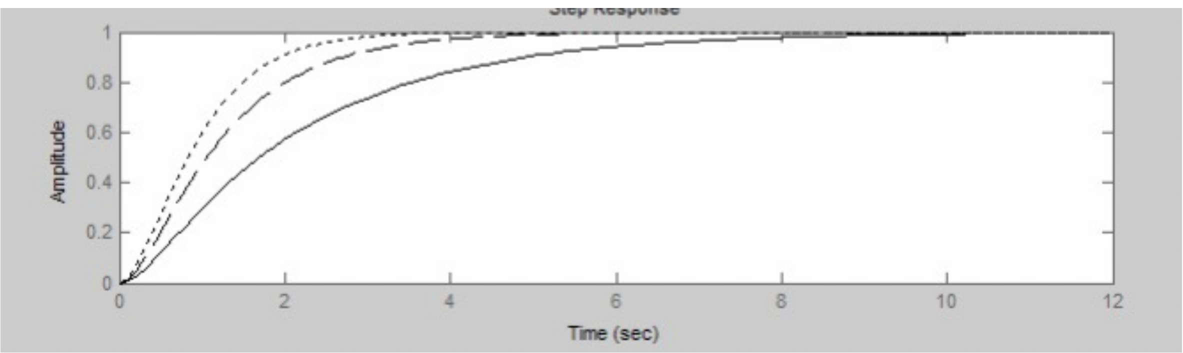
$$\frac{1}{s}$$

$$Y(s) = H(s)U(s)$$

$$y(t) = h(t) * 1(t)$$

$$= K[\frac{1}{p_1 p_2} + \frac{1}{p_1(p_1 - p_2)}e^{p_1 t} + \frac{1}{p_2(p_2 - p_1)}e^{p_2 t}]$$

$$= C_1 + p_2 C_2 e^{p_1 t} - p_1 C_2 e^{p_2 t}$$



Overdamped

Critically damped

$$H(s) = \frac{K}{(s-p_1)(s-p_2)} = K\frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$p_1, p_2 = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

$\xi = 1$ $\quad\quad$ $p_1 = p_2 = -\omega_n$

I'll omit the details and jump to the conclusion directly

$$h(t) = K\omega_n^2 t e^{-\omega_n t}$$

$y(t) = K[1 - (1 + \omega_n t)e^{\omega_n t}]$ Step response

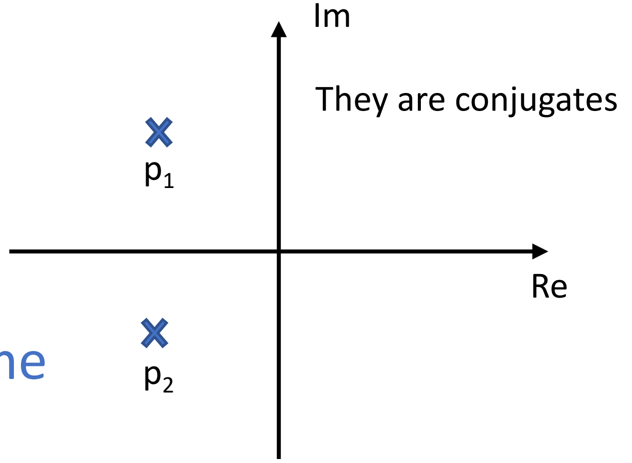
The shape is similar to the overdamped response

Underdamped response

$$H(s) = \frac{K}{(s-p_1)(s-p_2)} = K\frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$p_1, p_2 = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

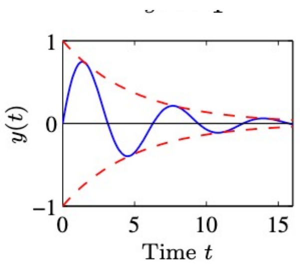
$$h(t) = K[\frac{1}{p_1 - p_2}e^{p_1 t} + \frac{1}{p_2 - p_1}e^{p_2 t}]$$



Let's consider a simplest case, p_1 and p_2 are complex numbers

$0 < \xi < 1$ What will happen to the transfer function? It's the same

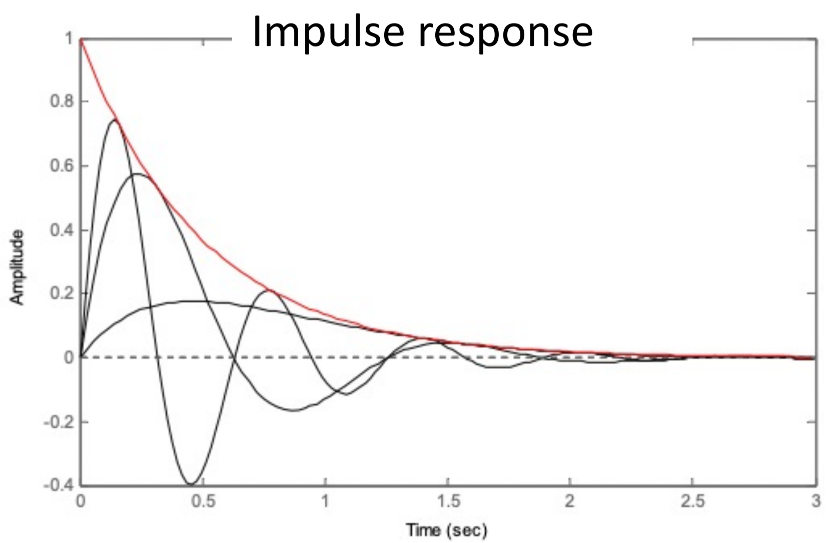
What's the shape of $e^{p_1 t}, e^{p_2 t}$?



$$p_1 - p_2 = 2\omega j$$

$$e^{p_1 t} - e^{p_2 t} = 2j\sin(\omega t) \times e^{-\sigma t}$$

$$h(t) = \frac{\sigma^2 + \omega^2}{\omega}e^{-\sigma t}\sin(\omega t)1(t)$$

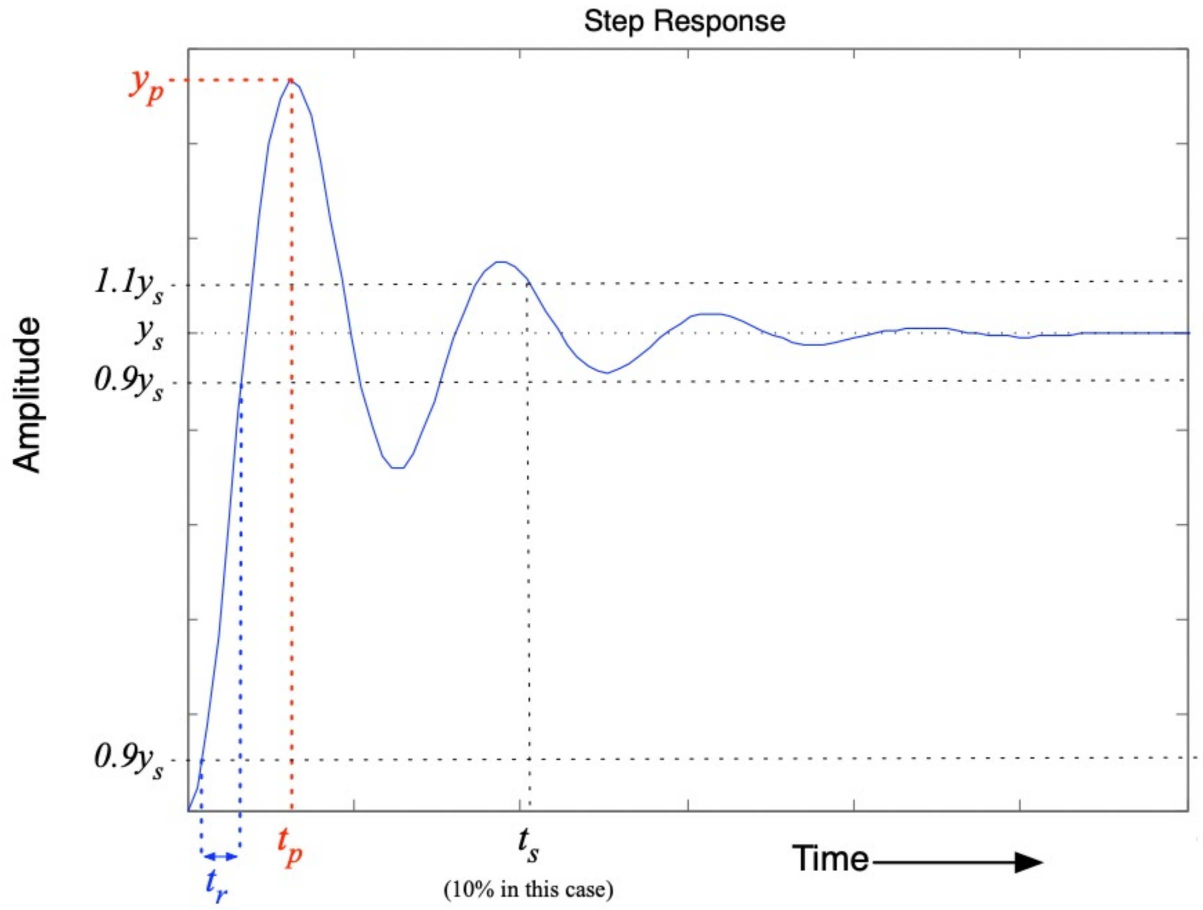


Frequency
Decay rates

Underdamped system- step response

$$h(t) = \frac{\sigma^2 + \omega^2}{\omega} e^{-\sigma t} \sin(\omega t) 1(t)$$

$$y(t) = K\left(1 - \frac{1}{\sqrt{1-\xi^2}} e^{-\xi \omega_n t} \sin\left(\omega_n \sqrt{1-\xi^2} t + \phi\right)\right) \quad = C_1 + C_2(e^{p_1 t} - e^{p_2 t})$$



Steady state value y_s: the value that y(t) goes to as t->inf

10%-90% rise time: the amount of time it takes to go from $0.1y_s$ to $0.9y_x$

Maximum percept overshoot

$$M_p = 100 \frac{|y_p - y_s|}{|y_s|}$$

n% settling time: the smallest time t_s so that for all $t>t_s$

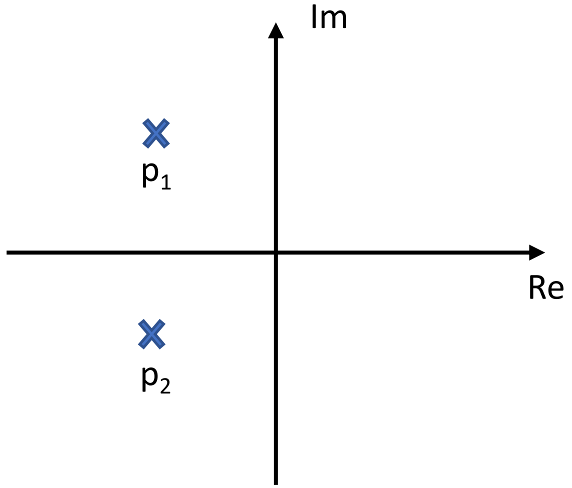
$$|y(t) - y_s| < \frac{n|y_s|}{100}$$

Rethink the solution

$$H(s) = \frac{K}{(s-p_1)(s-p_2)} = K\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} \qquad p_1, p_2 = -\xi\omega_n \pm \omega_n\sqrt{\xi^2-1}$$

$$h(t) = K[\frac{1}{p_1-p_2}e^{p_1 t} + \frac{1}{p_2-p_1}e^{p_2 t}] \qquad y(t) = C_1 + p_2 C_2 e^{p_1 t} - p_1 C_2 e^{p_2 t}$$

$$h(t) = \frac{\sigma^2+\omega^2}{\omega}e^{-\sigma t}\sin(\omega t)1(t) \qquad y(t) = C_1 + C_2(e^{p_1 t} - e^{p_2 t})$$



The impulse response is always decided by the poles! (also the oscillating part of the step response)

They are made of the *modes*