16-299 Intro to Feedback Control Systems Spring 2022

Frequency Domain Analysis (2)

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Recap of frequency lecture 1

- Laplace transform

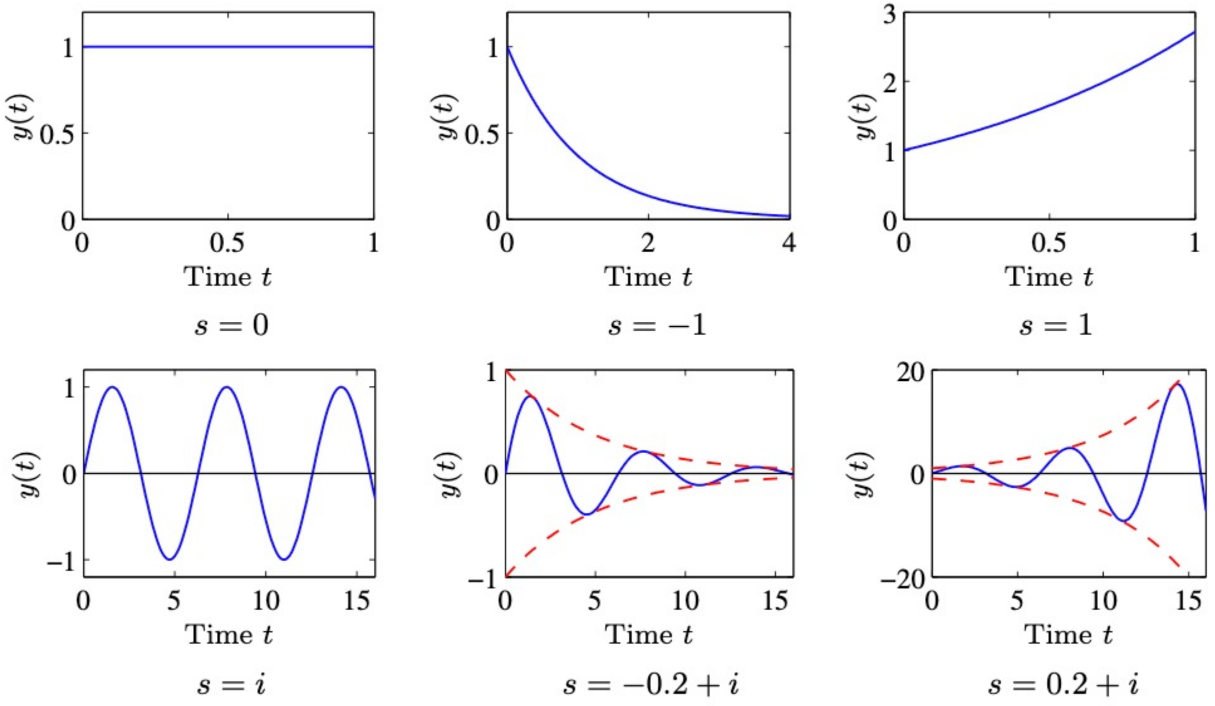
$$F(s) = \mathcal{L}\left[f(t)\right] = \int_{0_-}^{\infty} f(t)e^{-st}dt \qquad f(t) = \mathcal{L}^{-1}\left[F(s)\right] = \frac{1}{2\pi j}\int e^{st}F(s)ds$$

How to understand Laplace transform?

What's $\mathcal{L}\left[\frac{d}{dt}f(t)\right]$? $\quad sF(s) - f(0)$

What's $\mathcal{L}\left[\frac{d^n}{dt^n}f(t)\right]$?

$$s^nF(s) - \sum_{k=1}^{n} s^{n-k} \left.\frac{d^{k-1}}{dt^{k-1}}f(t)\right|_{t=0_-}$$



Recap of frequency lecture 1

- Transfer function $Y(s) = \underline{H(s)}U(s)$

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_n y = b_0 \frac{d^m u}{dt^m} + b_1 \frac{d^{m-1} u}{dt^{m-1}} + \cdots + b_m u, \rightarrow \sum_{i=0}^{n} a_i s^i Y(s) = \sum_{i=0}^{m} b_i s^i U(s)$$

$$H(s) = \frac{B(s)}{A(s)}$$

$H(i\omega) \rightarrow$

$r = \|H(i\omega)\|$ gain

$\phi = \angle H(i\omega)$ Phase shift

$$y(t) = \sum_k \|H(i\omega_k)\| A_k \sin(\omega_k t + \phi_k + \angle H(i\omega_k))$$

Transfer function and State space

- Transfer function provides a way to get state space representations of a system.
- Let's look at an example:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t),$$

$\xrightarrow{\mathcal{L}}$

$$sX(s) = AX(s) + BU(s)$$

$$Y(s) = CX(s).$$

We want to get H: u- > y

$$(sI - A)X(s) = BU(s)$$

$$X(s) = (sI - A)^{-1}BU(s),$$

$$Y(s) = C(sI - A)^{-1}B\,U(s)$$

$$H(s) = C(sI - A)^{-1}B$$

Transfer function and State space

State space (A, B, C)

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t),$$

Transfer function

$$H(s) = C(sI - A)^{-1}B.$$

?

Realization

(A, B, C) is a realization

A realization only exist when the transfer function is strictly casual $n > m$

Then there are infinite number of realizations

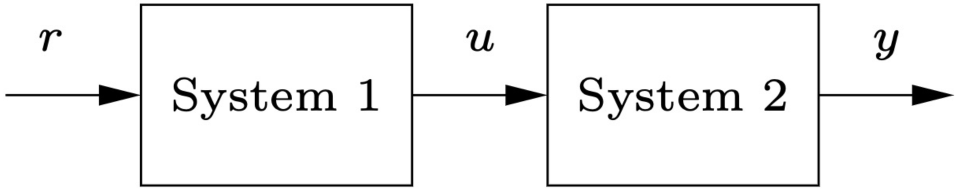
Why?

Using $z = Tx$, instead of x. What is z->y? What is (u, z)-> dz/dt?

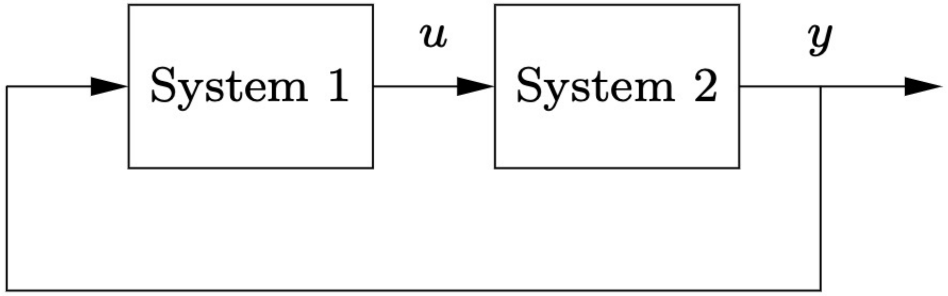
Block diagrams

- A typical way to represent a control system

Open loop

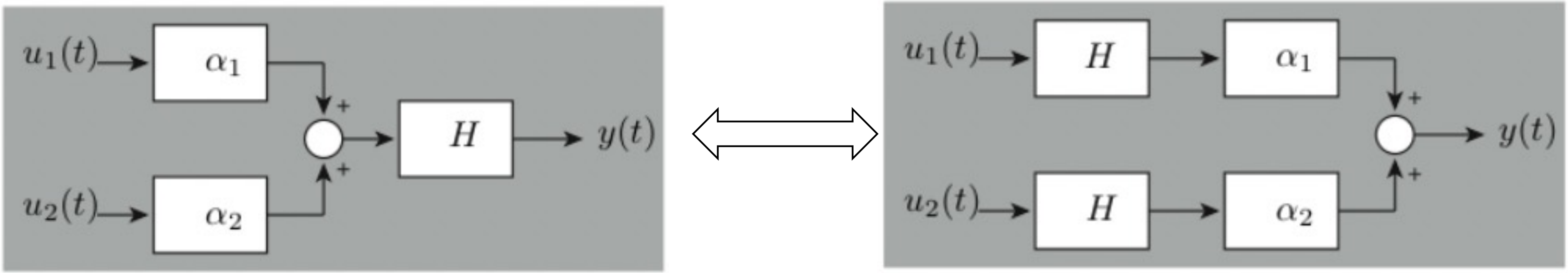


Closed loop



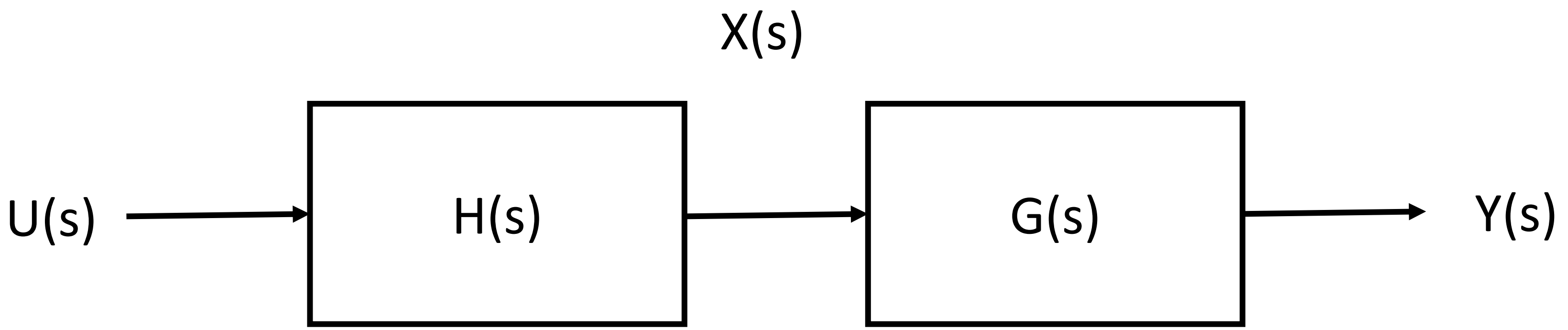
The system/operation could be described in either temporal domain or frequency domains

Linearity in block diagrams



The linearity exists in both time domain and frequency domain

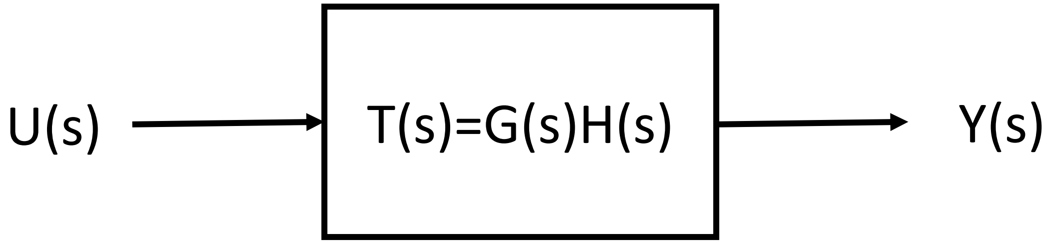
Block in Series



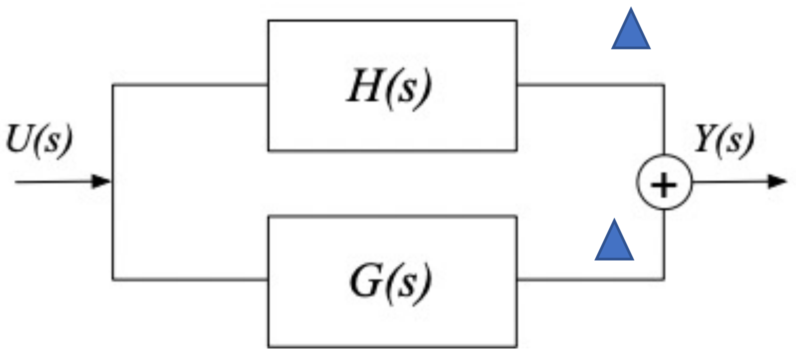
What is Y(s)?

X(s) = H(s)U(s)

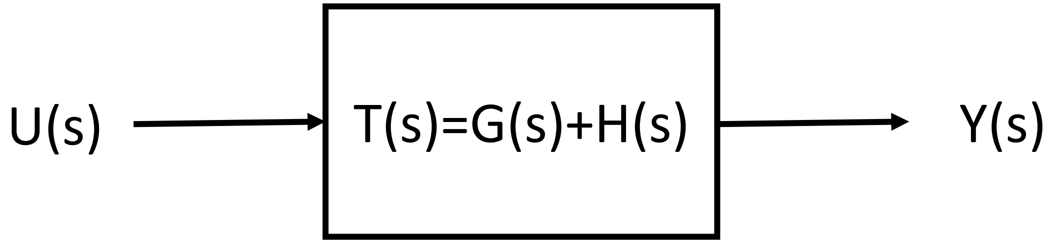
Y(s) = G(s)X(s) = G(s)H(s)U(s)



Block in parallel

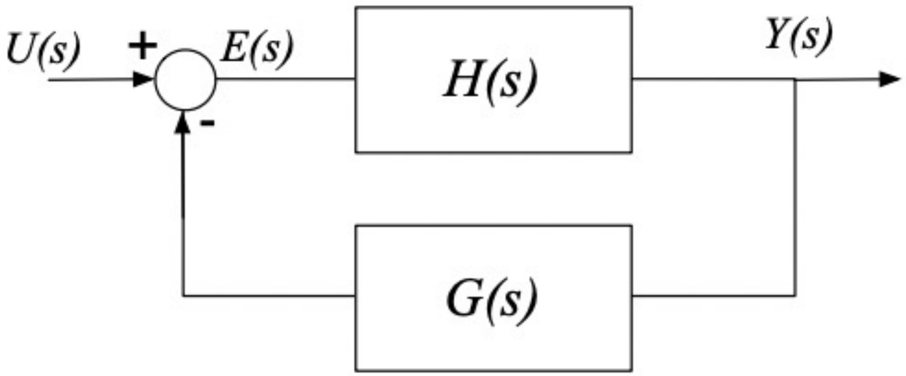


What is Y(s)?



Feedback

What is Y(s)?



$$Y(s) = H(s)E(s)$$

$$E(s) = U(s) - G(s)Y(s)$$

$$Y(s) = H(s)(U(s) - G(s)Y(s))$$

$$Y = \frac{H}{1+GH}U$$

So the closed loop transfer function is $T(s) = \dfrac{H(s)}{1+G(s)H(s)}$

Zero frequency gain

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_n y = b_0 \frac{d^m u}{dt^m} + b_1 \frac{d^{m-1}u}{dt^{m-1}} + \cdots + b_m u, \qquad Y(s) = H(s)U(s)$$

Zero frequency gain: $H(0)$ What does it mean?

Recap: $y(t) = \sum_k \|H(i\omega_k)\| A_k \sin(\omega_k t + \phi_k + \angle H(i\omega_k))$

$$\longrightarrow H(0) = \frac{y_0}{u_0} \qquad = \frac{a_n}{b_m}$$

Why?

The response of the constant component of the signal

Poles and zeros

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_n y = b_0 \frac{d^m u}{dt^m} + b_1 \frac{d^{m-1} u}{dt^{m-1}} + \cdots + b_m u, \qquad Y(s) = H(s)U(s)$$

Assuming $H(s) = \frac{b(s)}{a(s)}$ The roots of the polynomial a(s) is called the poles of the system, and roots of b(s) are called zeros of the system

What will happen when s=q, while q is one of the zeros of the system?

$H(s) = 0 \longrightarrow Y(s) = 0$ The output corresponding to $u(t) = e^{st}$ is zero

The signal is blocked

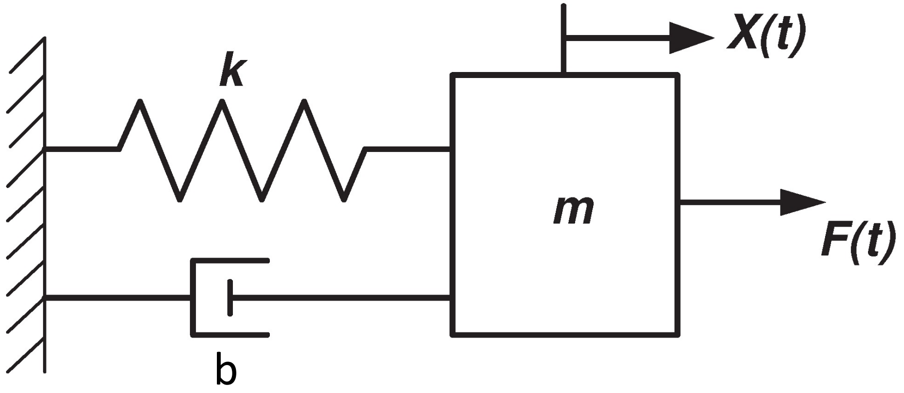
What will happen when s=p, while p is one of the poles of the system?

$y(t) = e^{pt}$ is a solution of the differential equation when u=0

u=0, U(s)=0 for all s, what is Y(s)?

Why do we care the solution of u=0?

Poles and zeros



$$m\frac{d^2x(t)}{dt^2} + b\frac{dx(t)}{dt} + kx(t) = f(t)$$

What is x(t) when f(t) = 0?

What is x(t) when f(t) = 0 + f'(t)?

What is x(t) when f(t) = 2 * 0 + f'(t)?

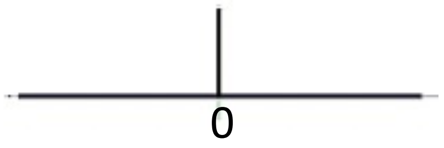
The output x is a combination of a solution to f'(t) and arbitrary combination of the solution to f(t)=0.

A pole p corresponds to a mode of the system with corresponding modal solution e^{pt}

The unforced motion of the system after an arbitrary excitation is a weighted sum of modes.

Stability

- What is stability?
- Thinking of the impulse response
- There are only two possible responses



When $t \to \infty$

$$y(t) \to 0$$ The system is *asymptotically stable*

$$y(t) \nrightarrow 0$$ The system is *unstable*

Why?

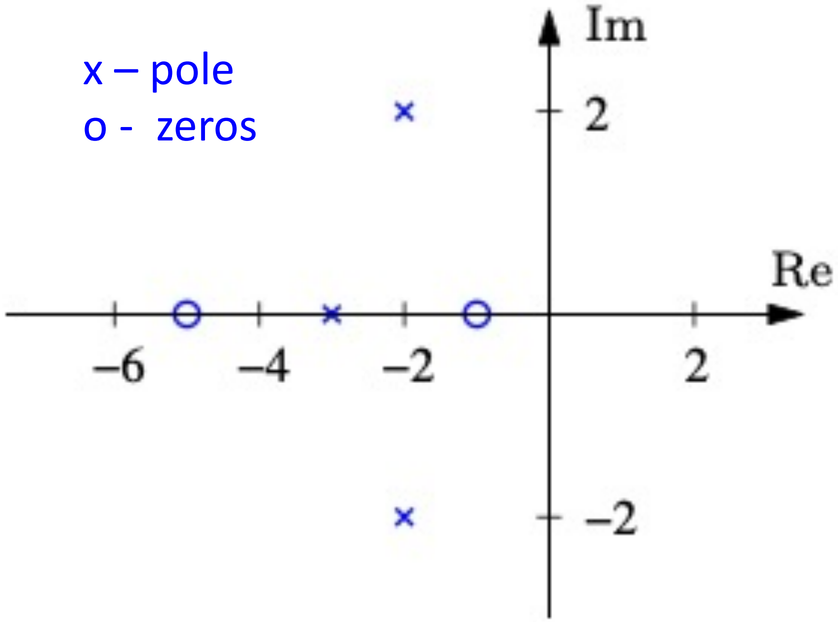
y(t) will explode since it's the sum of infinite response to impulse input

Stability of transfer functions

$$Y(s) = H(s)U(s) \qquad H(s) = \frac{b(s)}{a(s)}$$

- A system defined by a transfer function H(s) is stable if and only if all of the poles of H(s) have negative real part. Such system are said to be *Hurwitz*

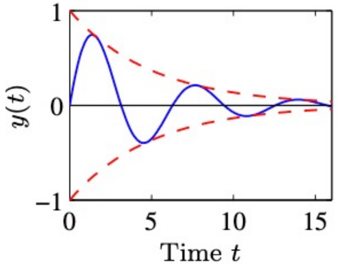
x – pole
o - zeros



$$y(t) = A_1 e^{-p_1 t} + A_1 e^{-p_1' t} + A_2 e^{-p_2 t} + y'(t)$$

A1 and A2 are arbitrary numbers

What if real(p_n)<0?



What if real(p_n)>0?