16-299 Intro to Feedback Control Systems Spring 2022

Frequency Domain Analysis

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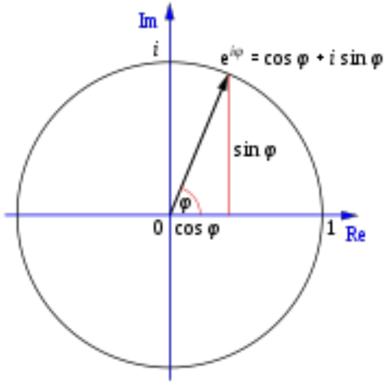
How to go from temporal domain to frequency domain?

Fourier Transform

Decompose the original signal to sinusoidal functions

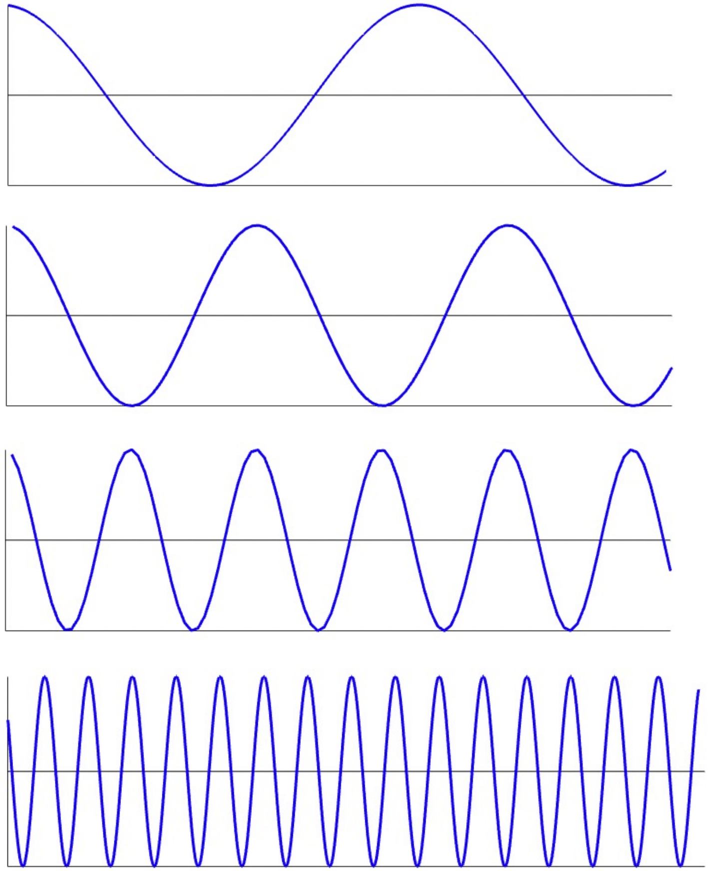
$$F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-j\omega x} dx$$

Function in temporal domain



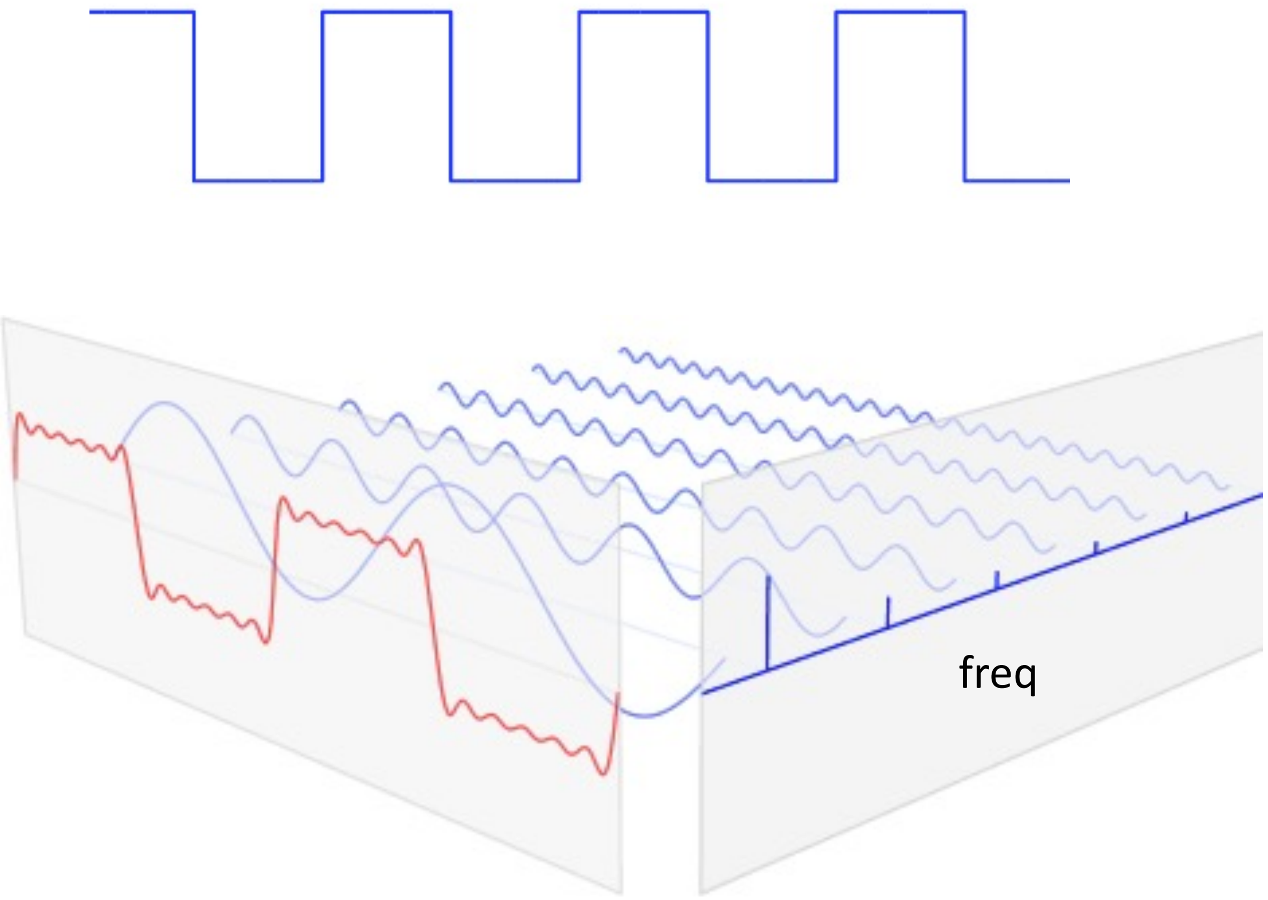
$$e^{j\phi} = \cos\phi + j\sin\phi$$

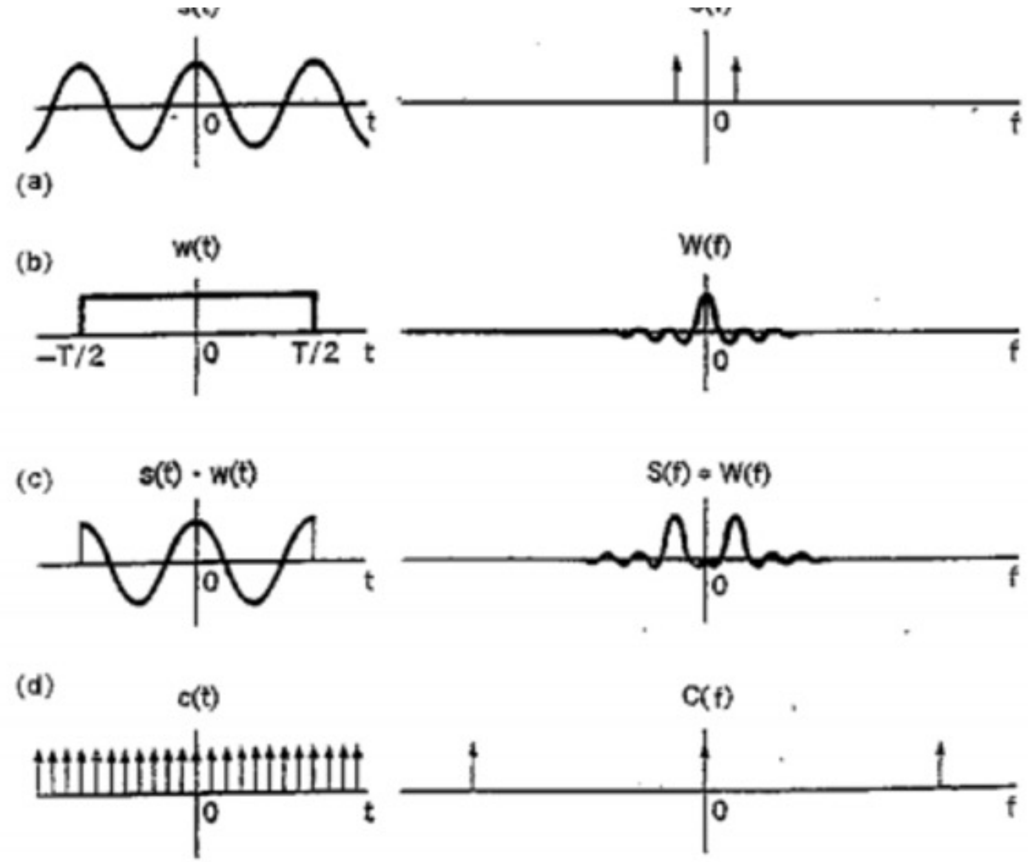
$$e^{j\phi} + e^{-j\phi} = 2cos\phi$$



For different frequencies (omega)

Examples of Fourier Transform





Inverse Fourier Transform

- Going back to temporal domain from frequency domain

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega} d\omega$$

$$F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-j\omega x} dx$$

Fourier transform

Laplace transform

- "An extended version of Fourier Transform"

$$F(s) = \mathcal{L}\left[f(t)\right] = \int_{0_-}^{\infty} f(t)e^{-st}dt$$

s is a complex number $s = \sigma + j\omega$

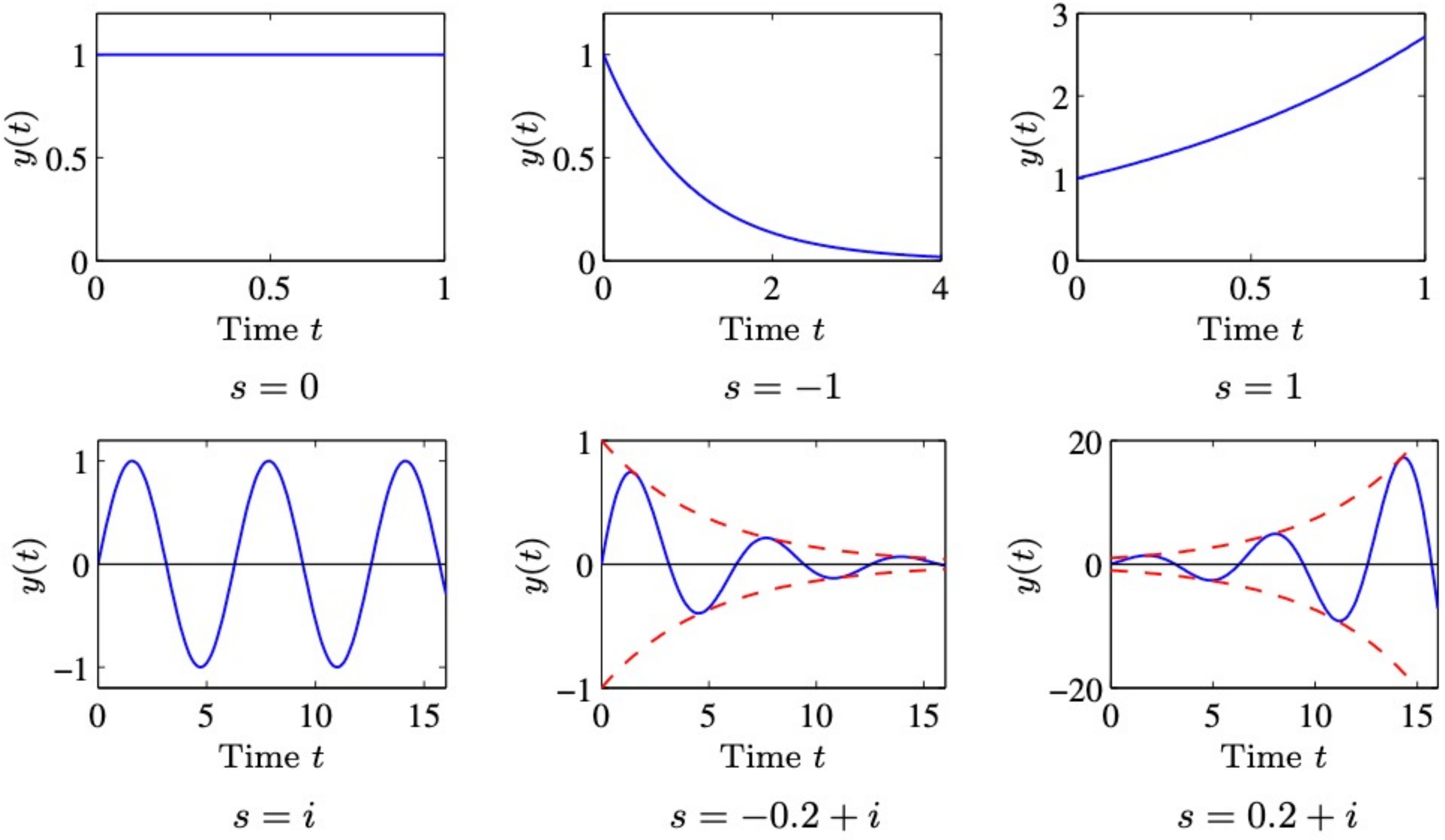
When $s = j\omega$, the Laplace transform becomes Fourier transform

For simplicity reasons, we denote $Y(s) = \mathcal{L}\left[y(t)\right]$ and sometime simplify it as Y

- Inverse Laplace transform

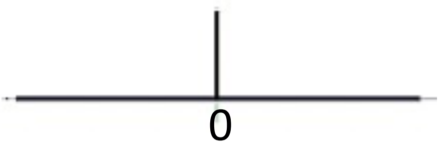
$$f(t) = \mathcal{L}^{-1}\left[F(s)\right] = \frac{1}{2\pi j}\int e^{st}F(s)ds$$

Examples of exponential signals e^{st}



Examples of Laplace transform

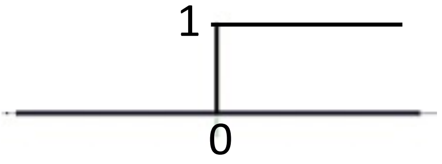
- Unit impulse function



$$\int_{-\epsilon}^{\epsilon} v(t)\delta(t) = v(0)$$

$$\mathcal{L}\left[\delta(t)\right] = \int_{-\epsilon}^{\epsilon} \delta(t)e^{-st}dt = 1$$

- Unit step function



$$u(t) = \begin{cases} 0 & \text{if} \quad t < 0 \\ 1 & \text{if} \quad t \geq 0 \end{cases}$$

$$U(s) = \mathcal{L}[u(t)] = \int_{0_-}^{\infty} e^{-st}dt$$

$$= \left[-\frac{1}{s}e^{-st}\right]_{0_-}^{\infty} = 0 - (-\frac{1}{s})$$

$$= \frac{1}{s}$$

Linear combination of two functions

Let $f(t) = \alpha_1 f_1(t) + \alpha_2 f_2(t)$ What is $F(s)$?

$$F(s) = \int_{0_-}^{\infty} (\alpha_1 f_1(t) + \alpha_1 f_2(t)) e^{-st} dt$$

$$= \alpha_1 \int_{0_-}^{\infty} f_1(t) e^{-st} dt + \alpha_2 \int_{0_-}^{\infty} f_2(t) e^{-st} dt$$

$$= \alpha_1 F_1(s) + \alpha_2 F_2(s).$$

The Laplace transform is a linear operator!

Integration of a function

$$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \int_0^\infty \left[\int_0^t f(\tau)d\tau\right] e^{-st}dt$$

$$= \left[\int_0^t f(\tau)d\tau \times -\frac{1}{s}e^{-st}\right]_0^\infty - \int_0^\infty -\frac{1}{s}e^{-st} \times f(t)dt$$

$$= \quad 0 \quad + \frac{1}{s}\int_0^\infty e^{-st} f(t)dt$$

$$= \frac{1}{s}F(s)$$

Recall that the integration by parts formula:

$$\int u dv = uv - \int v du.$$

We make $u = \int_0^t f(\tau)d\tau$ so $du = f(\tau)d\tau$

We make $dv = e^{-st}dt$ so $v = -\frac{1}{s}e^{-st}$

Derivatives of a function

Recall that the integration by parts formula:

$$\int udv = uv - \int vdu.$$

$$\mathcal{L}\left[\frac{d}{dt}f(t)\right] = \int_{0_-}^{\infty}\left(\frac{d}{dt}f(t)\right)e^{-st}dt.$$

$$(\frac{d}{dt}f(t))e^{-st}dt \longrightarrow \frac{d}{dt}f(t)dt \times e^{-st} \longrightarrow e^{-st}df(t)$$

$$\int_0^{\infty}(\frac{d}{dt}f(t))e^{-st}dt = \int_0^{\infty}e^{-st}df(t) = [f(t)e^{-st}]_0^{\infty} - \int_0^{\infty}f(t)de^{-st}$$

$$= [0 - f(0)] - \int_0^{\infty}f(t) \times (-se^{-st})dt$$

$$= -f(0) + s\int_0^{\infty}f(t)e^{-st}dt$$

$F(s)$

$$\mathcal{L}\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0_-).$$

Derivatives of a function

$$\mathcal{L}\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0_-).$$

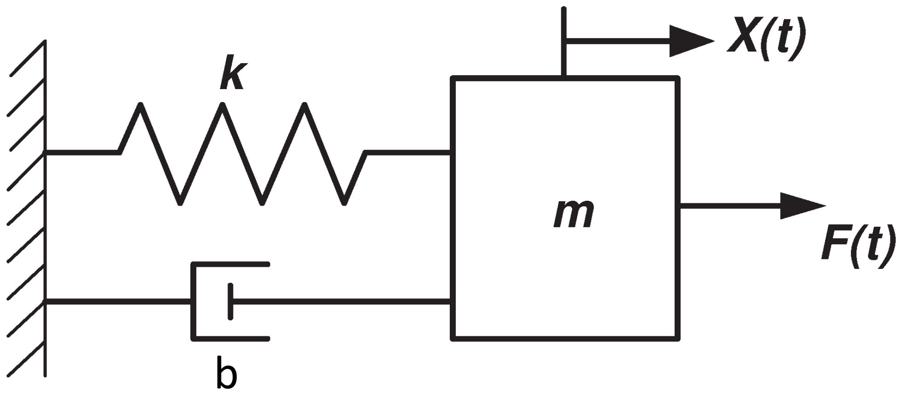
This is important, because it makes differential equations much easier!

How about higher derivatives?

$$\mathcal{L}\left[\frac{d^2}{dt^2}f(t)\right] = s^2F(s) - sf(0_-) - \left.\frac{d}{dt}f(t)\right|_{t=0_-}.$$

$$\mathcal{L}\left[\frac{d^n}{dt^n}f(t)\right] = s^nF(s) - \sum_{k=1}^{n} s^{n-k}\left.\frac{d^{k-1}}{dt^{k-1}}f(t)\right|_{t=0_-}.$$

Differential equations – the spring-damper model as an example



$$\mathcal{L}\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0_-).$$

$$\mathcal{L}\left[\frac{d^2}{dt^2}f(t)\right] = s^2F(s) - sf(0_-) - \left.\frac{d}{dt}f(t)\right|_{t=0_-}.$$

$$m\frac{d^2x(t)}{dt^2} + b\frac{dx(t)}{dt} + kx(t) = f(t)$$

Doing a Laplace transform

For simplification reasons, we assume x(0)=0, x'(0)=0

$$ms^2X(s) + bsX(s) + kX(s) = F(s)$$

$$(ms^2 + bs + k)X(s) = F(s)$$

$$X(s) = \frac{1}{ms^2 + bs + k}F(s)$$

Then we can perform inverse Laplace transform to get x(t)

Systems built on differential equations

- In many practical situations, the input/output behavior of a system can be modeled by a linear differential equation

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_n y = b_0 \frac{d^m u}{dt^m} + b_1 \frac{d^{m-1} u}{dt^{m-1}} + \cdots + b_m u,$$

u is the input, y is the output. Coefficients a_k and b_k are real numbers

$$\mathcal{L}\left[\sum_{i=0}^{n} a_i \frac{d^i}{dt^i} y(t)\right] = \mathcal{L}\left[\sum_{i=0}^{m} b_i \frac{d^i}{dt^i} u(t)\right] \rightarrow \sum_{i=0}^{n} a_i \mathcal{L}\left[\frac{d^i}{dt^i} y(t)\right] = \sum_{i=0}^{m} b_i \mathcal{L}\left[\frac{d^i}{dt^i} u(t)\right] \rightarrow \sum_{i=0}^{n} a_i s^i Y(s) = \sum_{i=0}^{m} b_i s^i U(s)$$

$$\underbrace{\left(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_2 s^2 + a_1 s + a_0\right)}_{\triangleq A(s)} Y(s) = \underbrace{\left(b_m s^m + b_{m-1} s^{m-1} + \cdots + b_2 s^2 + b_1 s + b_0\right)}_{\triangleq B(s)} U(s)$$

We define the transfer function $H(s) = \frac{B(s)}{A(s)}$

Then we have $Y(s) = H(s)U(s)$

Transfer function

- A core concept in classical controls

$$H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} = \frac{B(s)}{A(s)}$$

- We can think of H(s) as an operator that maps inputs to outputs
  - Either H: U(s)-> Y(s)
  - Or H: u(t) -> y(t)
- Typically, m<=n, and we call the transfer function *causal*
- H is a linear operator

$$H(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 H(u_1) + \alpha_2 H(u_2)$$

- H is a time invariant operator:

$$\text{if } H(u(t)) = y(t), \text{ then } H(u(t-\tau)) = y(t-\tau)$$

} Linear and time invariant (LTI) systems

Transfer function

- The LTI nature of transfer functions:
- Let H: u-> y be a LTI operator, and $u(t) = A_{in} \sin(\omega t + \phi_{in})$.
- Then the output y(t) is a pure sinusoid *of the same frequency*

$$y(t) = A_\omega A_{in} \sin(\omega t + \phi_{in} + \phi_\omega)$$

Why?

Transfer function

$$Y(s) = H(s)U(s)$$

- Consider a case: let $s = i\omega$

$H(i\omega)$ Is complex

$r = \|H(i\omega)\|$ magnitude → gain

$\phi = \angle H(i\omega)$ angle → Phase shift

In other words, if the input is $u(t) = A_{in}\sin(\omega t + \phi_{in})$

The output is $y(t) = \|H(i\omega)\| A_{in} \sin(\omega t + \phi_{in} + \angle H(i\omega))$

Considering the input is a sum of sinusoids $u(t) = \sum_k A_k \sin(\omega_k t + \phi_k)$, what's the output?

$$y(t) = \sum_k \|H(i\omega_k)\| A_k \sin(\omega_k t + \phi_k + \angle H(i\omega_k))$$