15-354: Computational Discrete Math K. Sutner

Assignment 9 Due: November 17, 2009

Problem 1: Boolean Circuits (30 pts.)

Background

Given a circuit that computes some particular Boolean function one can easily modify input and output channels to obtain other Boolean functions. We have seen in class how to count Boolean functions modulo the equivalence induced by inverting some inputs. Here are some similar counting problems.

Task

1. Count the number of equivalent Boolean functions on k inputs when both inputs and the output can be inverted.
2. Count the number of equivalent Boolean functions on k inputs if the inputs can be rotated. So if the rotation is by one place to the left we have

$$f(x_1, x_2, \ldots, x_k) = g(x_2, x_3, \ldots, x_k, x_1)$$

3. Count the number of equivalent Boolean functions on k inputs when inputs can be arbitrarily permuted:

$$f(x_1, x_2, \ldots, x_k) = g(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(k)})$$

where π is any permutation of $[k]$.

Comment

For the last two problems dont try to come up with simple closed-form solutions; they dont exist. But try to write down the answer concisely and elegantly.

Problem 2: Counting Cubes (40 pts.)

Background

We have seen in class that the dihedral group D_4 describes all rigid motions that move a square back to itself, including reflections. This and similar dihedral groups together with the Polya machinery are very helpful to solve a variety of "two-dimensional" counting problems. There is an obvious three-dimensional analogue based on the group G of rigid motions that moves a cube back to itself. There is a natural subgroup H that excludes reflections (so there are only rotations to deal with).

Task

A. Find a (small) set of generators for H. What is the size of H?

B. Find a (small) set of generators for G. What is the size of G?

C. How do the group elements act as permutations on the set of vertices of a cube?

D. Determine the cycle index polynomial of H and G.

Extra Credit: Find a minimal set of generators and describe the equations that hold between these generators analogous to our description of D_n as $a^n = 1$, $ab = ba^{n-1}$.

Comment

Unless you have perfect geometric intuition it's probably a good idea to use a physical model of a cube to find the rotations. A Rubik's Cube should work fine.

Stay away from the pattern inventory, without computer algebra that gets to be quite tedious.

Problem 3: BITs (30 pts.)

Background

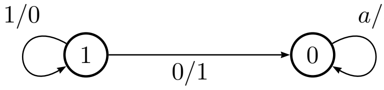
A *binary invertible transducer (BIT)* is a finite state machine that has transitions

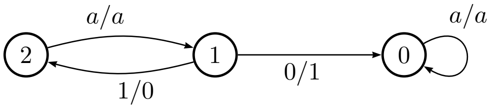
$$p \xrightarrow{a/b} q$$

where $a, b \in \mathbf{2}$. Moreover, for each state p, there are two outgoing transitions labeled

$$p \xrightarrow{a/a} q_a \qquad \text{or} \qquad p \xrightarrow{a/\overline{a}} q_a,$$

p is referred to as *copy* and *toggle* state, respectively. Here are two examples $\mathcal{A}$ and $\mathcal{B}$:





Thus, state 1 is the only toggle state in both $\mathcal{A}$ and $\mathcal{B}$, the other states are all copy states.

We obtain maps $T_p : \mathbf{2}^\star \to \mathbf{2}^\star$ by selecting some state p as initial state. These maps are always permutations, so we can consider the group G generated by the maps T_p, $p \in Q$. Given a word $x \in \mathbf{2}^n$ we can define its orbit $T_p^\star(x) \subseteq \mathbf{2}^n$ under T_p.

Task

A. Suppose M is an arbitrary BIT. Explain how to construct another BIT M' so that M' defines all the inverse transductions T_p^{-1} of M.

B. For the 2-state transducer $\mathcal{A}$ above, determine the orbits of all words under T_1.

C. Describe the group G associated with $\mathcal{A}$.

D. For the 3-state transducer $\mathcal{B}$ above, determine the orbits of all words under T_1 and T_2.

E. Describe the group G associated with $\mathcal{A}$.

Comment

$\mathcal{B}$ is very closely related to $\mathcal{A}$, very closely.