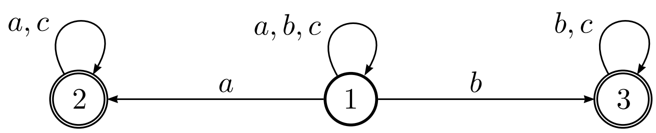
15-354: Computational Discrete Math	K. Sutner
Assignment 8	Due: November 10, 2010

Problem 1: Determinization of Büchi Automata (40 pts.)

Background

Below is the natural Büchi automaton $\mathcal{A}$ for the language $L \subseteq \{a, b, c\}^\omega$ of all words that contain either at least one a but finitely many b's or at least one b but finitely many a's.



Here $I = \{1\}$ and $F = \{2, 3\}$.

Task

A. Write a "regular expression" with ω-terms for this language.

B. Run Safra's algorithm on $\mathcal{A}$ to obtain a Rabin automaton for L.

C. Construct a Büchi automaton for the complement of L.

Comment

For the last part, make sure to draw a picture. It's not at all bad if you pick the right layout. Step 1 is to draw a nice picture for the Rabin automaton in part (B). Think linear.

Problem 2: An Associative Operation (30 pts.)

Background

Many operations in algebra are clearly associative. However, in some cases associativity is far from obvious and requires a proof.

For this problem, write $[x]$ for the integer nearest to $x \in \mathbb{R}$. More precisely, let $[x] = \lfloor x + 1/2 \rfloor$ so that $[3.5] = 4$. Consider the map $\pi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by

$$\pi(x, y) = \begin{cases} x + y^2 & \text{if } x \neq \max(x, y), \\ x(x+1) + y & \text{otherwise.} \end{cases}$$

We can think of the last map is a bijection between $\mathbb{N}$ and the first quadrant in the (discrete) plane $\mathbb{Z} \times \mathbb{Z}$, see part (A). Here is a slightly different map: $\rho : \mathbb{N} \to \mathbb{N} \times \mathbb{Z}$

$$\rho(x) = ([\sqrt{x}], x - [\sqrt{x}]^2)$$

Let $W \subseteq \mathbb{N} \times \mathbb{Z}$ be the range of ρ.

A. Show that π is a bijection and determine the unpairing functions.

B. Determine W, the range of ρ and show that ρ is a bijection between $\mathbb{N}$ and W.

C. Show that the binary operation on $\mathbb{N}$ defined by

$$a * b = a + b + 2[\sqrt{a}][\sqrt{b}]$$

is associative.

Comment

Part (A) is just a warm-up, but part (B) is useful for (C). The latter can also be handled by brute force, but that's a mess.

Problem 3: Generating Permutations and Functions (30 pts.)

Background

We have seen that all permutations of $[n]$ can be obtained from the cyclic shift $(2, 3, \dots, n, 1)$ and the transposition $(2, 1, 3, \dots, n)$ by composition (we are using one-line notation). Needless to say, there are many other generating sets (e.g., the set of all transpositions) and there are quite a few of size 2.

Task

A. Describe some generating sets of size 2 for the group of all permutations of $[n]$. Are there many of these (for some reasonable sense of many)?

B. Characterize all possible generating sets of size 3 for the monoid of all functions $[n] \to [n]$ assuming all sets of generators of size 2 for the symmetric group are known.

C. Find a small set of generators for the alternating group on n points. You might want to find an analogue to the fact that the full symmetric group is generated by transpositions first.

Comment

As always, "characterize" means that you have to find some condition that holds if, and only if, f, g and h are generators for the monoid of all functions. Try to find a simple, elegant condition. Then prove that your condition really works.

Note that the corresponding problem for general relations (under relational composition) is much more difficult: there is no fixed size set of generators, and the proof is hard.