

**Solution: Definite Automata****Part A:** Third from the end

Consider the de Bruijn automaton of rank 3 over  $\{a, b\}$ . If we pick  $bbb$  as the initial state and all states  $axy$ ,  $x, y \in \{a, b\}$  as final we get a DFA  $M$  that accepts  $L_{a,-3}$ .

Moreover,  $M$  is clearly definite with  $r = 3$ .

**Part B:** Definite Languages

Let  $r$  be the maximum length of any word in  $A$  and consider the de Bruijn transition system of order  $r+1$  over  $\Gamma = \Sigma \cup \{\#\}$  where  $\# \notin \Sigma$ . Pick  $q_0 = (\#, \dots, \#)$  as the initial state and select as final any state that has a suffix in  $A$ .

Then for any word  $x$  of length at most  $r$  we have  $\delta(q_0, x) = \#^{r-|x|}x$  and otherwise  $\delta(q_0, x) = v$  where  $x = uv$  and  $|v| = r$ . But then  $M$  accepts precisely  $\Sigma^* A$ . and definiteness follows immediately.

Note that we can omit all transitions with label  $\#$  (we need a machine over  $\Sigma$ , not  $\Gamma$ ) and take only the part accessible from  $q_0$ .

**Part C:** Decision Algorithm

We may safely assume that  $M$  is an accessible DFA. Let  $M^2$  be the Cartesian product of  $M$  with itself. Note that  $M^2$  contains an isomorphic copy of  $M$  as the “diagonal,” all states of the form  $(p, p)$ .

For  $M$  to be definite we need all sufficiently long paths starting at any point  $(p, q)$  to wind up in the diagonal.

But this means that there is no non-trivial strongly connected component in  $M^2$  outside of the diagonal, all points outside of the diagonal are transient.

**Part D:** Time Complexity

The construction of  $M^2$  is quadratic in the worst case, but the decomposition into strongly connected components (using, say, Tarjan’s algorithm) is linear. So, the whole algorithm is quadratic.

**Solution: Right Quotients****Part A:** Boolean Ops

Consider union.

$$(L \cup K)/x = \{y \mid yx \in L \cup K\} = \{y \mid yx \in L\} \cup \{y \mid yx \in K\} = L/x \cup K/x.$$

The arguments is the same for the other operations.

**Part B: Decomposition**

Write  $Q^+ = \{q \mid \delta(p, w_q) \in F\}$  and  $Q^- = Q - Q^+$ .

First let  $x \in A$ . Then  $\delta(q_0, xw_q) \in F$  for all  $q \in Q^+$ , so  $x \in \bigcap_{q \in Q^+} L/w_q$ . By the same token,  $x \in \bigcap_{q \in Q^-} (\Sigma^* - L/w_q)$  so that  $x \in K$ .

For the opposite direction consider  $x \notin A$ . Then  $\delta(q_0, x) = q \neq p$  for some  $q$ . If  $q \in Q^+$  we have  $\delta(q_0, xw_q) \notin F$ . Likewise, for  $q \in Q^-$  we have  $\delta(q_0, xw_q) \in F$  and it follows that  $x \notin K$ .

**Part C: Boolean combination**

From the last part we have  $L = \bigcup_{p \in F} \bigcap_{q \neq p} R_q$ .

**Solution: The Un-Equal Language**

**Part A: Quotients  $L_2$**

The quotients of  $L_2$  organized by length of the dividing word are:

- 0  $L_2$
- 1  $(aab, aba, abb, baa, bba, bbb), (aaa, aab, abb, baa, bab, bba)$
- 2  $(ab, ba, bb), (aa, ba, bb), (aa, ab, bb), (aa, ab, ba)$
- 3  $(a), (b), (a, b)$
- 4  $(\epsilon)$
- 5  $\emptyset$

**Part B: Quotients**

Ignoring the sink, states are organized in layers, at layer  $\ell$  we have  $x^{-1}L_k = P \subseteq \{a, b\}^{2^{k-\ell}}$  where  $|x| = \ell$ . In particular for  $\ell = k$  we have  $u^{-1}L_k = \{a, b\}^k - \{u\}$  and the structure of the quotient automaton down to level  $k$  is a complete binary tree; the number of states in this part is  $2^{k+1} - 1$ .

The remainder of the machine has two kinds of states: those where a witness for inequality of  $u$  and  $v$  has already been found, and those where we are still waiting for such a witness.

Fix some  $u \in \{a, b\}^k$ . The first type of state is of the form  $\delta(q_0, ux)$  where  $x$  is not a prefix of  $u$  and has behavior  $\{a, b\}^{k-|x|}$ , where  $|x| \leq k$ . The second type is of the form  $\delta(q_0, ux)$  where  $x$  is a prefix of  $u$  and has behavior  $\{a, b\}^{k-|x|} - \{y\}$  where  $u = xy$ . But then there are  $k$  states of the first type, and  $2^k - 1$  states of the second type (these form another tree which is upside-down compared to the first, and has as root the sink of the DFA).

**Part C: State Complexity**

It follows from the analysis in part (B) that the state complexity of  $L_k$  is  $3 \cdot 2^k - n + 2$ .