CDM

Ordinals and Cardinals

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Battleplan

- Transfinite Counting
- Ordinals
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- Cardinals

Cantor's Ordinals

Cantor's Construction

Cantor's early work is in analysis. He was interested in exploring the limits of Fourier analysis, which lead him to consider the following types of sets of reals.

Definition 1. *Let $A \subseteq \mathbb{R}$ closed. A point $a \in \mathbb{R}$ is* isolated *(in A) if there exists a neighborhood of a that is disjoint from A. Otherwise a is a limit point. A is* perfect *if it has no isolated points.*

Perfect sets can be constructed from arbitrary closed sets by removing isolated points. Write

$$X' = \{ \, x \in X \mid x \text{ limitpoint} \, \}$$

Unfortunately, $A' \subseteq A$ may not be perfect, either. So we repeat the process: A'' , A''' and so on. More precisely, we form a sequence $A_0 = A$, $A_{n+1} = A'_n$ for all $n \in \mathbb{N}$.

Exercise 1. *Show that it may happen that A_n contains isolated points, for all n.*

Limit Stages

There is no guarantee that A_n will be perfect for any $n \in \mathbb{N}$.

How do we continue our pruning process beyond the first infinitely many stages?

It's a fair guess that we should consider $\bigcap A_n$ next. We use ω to denote this next stage in the process.

$$A_\omega = \bigcap_{n \in \mathbb{N}} A_n$$

The weary reader might expect at this point that A_ω is not necessarily perfect either. So we continue our pruning process with stages $\omega + 1$, $\omega + 2$, ... We get to a stage $\omega + \omega = 2\omega$, then 3ω, and even $\omega\omega = \omega^2$, and further to ω^3 and so on, ad nauseam.

This is just wild and wooly notation so far, we have not defined these stage objects, much less their arithmetic.

Exercise 2. *Why must this process ultimately end?*

Ordinals

We want to define a class On of stage objects, henceforth called *ordinals*. The conditions are

- There is a unique least ordinal, denoted 0.
- For every ordinal α there is a unique least larger ordinal, denoted $\alpha + 1$.
- For every increasing sequence (α_i) there is a unique least upper bound, denoted $\sup \alpha_i$.

We are cheating a bit, of course: we also need to define an order relation $<$ on On.

Note that we can identify $\mathbb{N}$ with an initial segment of this order.

There is another problem: sequences are usually indexed by $\mathbb{N}$, but we need much larger index sets. To get around this, we can demand that $\sup X$ exists for any set $X \subseteq On$.

Carful, On is not a set but a proper class, so there is no contraction between the two conditions (On does not have a largest element).

Successor versus Limit

Disregarding 0, there are two types of ordinals:

- Successor ordinals of the form $\alpha + 1$. These indicate then next step immediately following step α.
- Limit ordinals of the form $\lambda = \sup X$. These indicate the collection of all prior steps into one super-step.

For example, for any limit ordinal λ we have

$$\alpha < \lambda \text{ implies } \alpha + 1 < \lambda.$$

It is customary to write ω for the smallest limit ordinal.

It is tempting to think of ω as just being $\mathbb{N}$, but it's helpful to have special notation for this number.

Well-Orderings

Definition 2.
A structure $\langle A, \lhd \rangle$ is a well-order *if $\lhd$ is a total order on A and every non-empty subset of A has a $\lhd$-minimal element.*

The classical example of a well-order is $\langle \mathbb{N}, < \rangle$. The integers and the positive rationals are the standard counterexample.

Lemma 1. *$\langle A, \lhd \rangle$ is a well-order if, and only if, there are no infinite descending chains in this order.*

In other words, we must not have a sequence

$$a_0 \rhd a_1 \rhd a_2 \rhd \ldots \rhd a_n \rhd \ldots$$

Exercise 3. *Prove the lemma (using intuitive set theory).*

Do They Exist?

A key question in the early development of set theory was whether every set can be well-ordered. Cantor thought this was "self-evident" because we construct the required order in stages. To well-order A construct a sequence a_α in stages

$$a_\alpha = \text{ pick some } x \in (A - \{\, x_\beta \mid \beta < \alpha \,\}$$

The problem is the "pick some" operations: A is an abstract set and there is no clear mechanism how this choice should be made.

In fact, try this even for a concrete set such as the reals, say $\mathbf{2}^{\mathbb{N}}$.

We need some high-powered axiom to make the selection of an element in an arbitrary, non-empty set possible: the Axiom of Choice.

Some Equivalences

The following principles are equivalent to the Axiom of Choice, but often easier to apply in concrete situations.

- The Well-Ordering Principle: every set can be well-ordered.
- Zorn's Lemma: every partial order in which every chain has an upper bound contains a maximal element.
- Hausdorff's Maximality Principle: every partial order has a maximal chain.
- Every equivalence relation has a set of representatives.

Zorn's Lemma was popularized in particular by Bourbaki.

We won't worry about foundational issues and simply assume either and all of the above.

Recursion and Termination

Well-orders are centrally important to termination of recursive procedures. Suppose some recursive function f when called on x makes a recursive call to y, where x and y belong to some ground set A.

If we can find a well-order $\lhd$ on A such that $y \lhd x$ then any call is always guaranteed to terminate: otherwise we would have an infinite descending chain.

As an example, consider the Ackermann function $A : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ defined by

$$\begin{aligned} A(0, y) &= y^+ \\ A(x^+, 0) &= A(x, 1) \\ A(x^+, y^+) &= A(x, A(x^+, y)) \end{aligned}$$

The standard proof for A being total is by induction on x and subinduction on y. This is a perfectly civilized approach, but it obscures some of the finer points.

Why Bother?

It may seem that any recursive definition along the lines of Ackermann's is automatically guaranteed to produce a well-defined, total function. Alas, that's most emphatically not the case.

$$f(0,0) = 0$$
$$f(x+1,y) = f(x,y+1)$$

$$g(x,0) = x+1$$
$$g(x,y+1) = g(x+1,y)$$
$$g(x+1,y+1) = g(x,g(x,y))$$

Exercise 4. *Explain what goes wrong in the definitions of f and g above.*

Well-Ordering $\mathbb{N} \times \mathbb{N}$

The ground set here is $A = \mathbb{N} \times \mathbb{N}$. There is a natural order on A, namely the lexicographic order:

$$(a, b) \prec (c, d) \iff (a < c) \vee (a = c \wedge b < d).$$

This order is a well-order and it is easy to see that whenever a call to Y is nested inside a call to X we have $Y \prec X$:

$(x, 0) \succ (x - 1, 1)$,

$(x, y) \succ (x, y - 1), (x - 1, z)$.

Thus termination is guaranteed.

Exercise 5. *Prove that $\langle A, \prec \rangle$ is indeed a well-order.*

Enumerating a Well-Order

Unlike arbitrary total orders, well-orders are always comparable in a very strict sense.

Suppose $\langle A, \lhd \rangle$ is a well-order. We can enumerate the elements of A by constructing a partial function $f : On \longrightarrow A$ defined by

$$f(0) = \min_{\lhd}(A)$$

$$f(\alpha + 1) = \min_{\lhd}(A - \{f(0), \ldots, f(\alpha)\})$$

$$f(\lambda) = \min_{\lhd}(A - \{\, f(\nu) \mid \nu < \lambda \,\}$$

Note the domain of f is an initial segment of On, so we get an order isomorphism

$$f : \{\, \alpha \in On \mid \alpha < \beta \,\} \longrightarrow A$$

Length of a Well-Order

The ordinal β is uniquely determined by A, so we can propose the following measure of the length of a well-order.

Definition 3. *The ordinal β such that that there is an order isomorphism from $\{ \alpha \in On \mid \alpha < \beta \}$ to A is the order type or length of A.*

Note that any two well-orders are comparable in the sense that one must be isomorphic to an initial segment of the other.

The length of a well-order is a successor ordinal if the order has a largest element, and a limit ordinal otherwise.

So, in a sense all well-orders reduce to initial segments of the ordinals. This fact is particularly useful since we can naturally define arithmetic of ordinals – which produces a handy notation system for well-orders.

Transfinite Induction

Transfinite Induction

We can extend the classical principle of induction on the naturals to induction on On.

Definition 4. *Let $\Phi(x)$ be some formula where x is supposed to range over ordinals. Φ is* inductive *if $\forall\, \beta < \alpha\, \Phi(\beta) \rightarrow \Phi(\alpha)$*

Theorem 1. *If $\Phi(x)$ is inductive then $\Phi(\alpha)$ holds for all $\alpha \in On$.*

Likewise, we can define functions on the ordinals by recursion. Here is the version with an additional set parameter.

Theorem 2. *von Neumann, 1923, 1928*
Given a function $F(x, y)$ defined on sets, there is a unique function $f : V \times On \rightarrow V$ defined by

$$f(x, \alpha) = F(x, \lambda z f(x, z) \downarrow \alpha).$$

In practice, is still often convenient to distinguish between arguments 0, successor ordinals and limit ordinals.

Example: An Order Isomorphism

Consider the two well-orderings $A = \langle \mathbb{N}^+ \times \mathbb{N}^+, < \rangle$ with the usual lexicographic order and $B = \langle \mathbb{N}^+, \prec \rangle$ defined as follows:

$$n \prec m \iff \nu_2(n) < \nu_2(b) \lor \nu_2(n) = \nu_2(b) \land n < m.$$

Here $\nu_2(x)$ is the highest power of 2 which devides x. Thus

$$1 \prec 3 \prec 5 \prec \ldots 2 \prec 6 \prec 10 \prec \ldots 4 \prec 12 \prec 20 \prec \ldots$$

The most concrete way to show that both orders are isomorphic is to give an explicit order-isomorphism. That's not too hard in this case: $F(a, b) = 2^a(2b - 1) - 1$.

Of course, we have to check that F is indeed a bijection and order-preserving.

Exercise 6. *Fill in the details in this argument.*

Using Induction

Another more abstract approch is to construct the order-isomorphism by induction simultaneously on both orders.

$$
\begin{aligned}
f_0 &= \emptyset \\
f_{\alpha+1} &= f_\alpha \cup \{(\min_<(A - \operatorname{dom} f_\alpha), \min_\prec(B - \operatorname{rg} f_\alpha)\} \\
f_\lambda &= \bigcup_{\alpha<\lambda} f_\alpha
\end{aligned}
$$

This is rather elegant. Note that it is trivial that $f = \bigcup f_\alpha$ is a partial order-isomorphism. Alas, one has to show that the two sets $A - \operatorname{dom} f_\alpha$ and $B - \operatorname{rg} f_\alpha$ become empty at the same stage (which happens to be ω^2).

Exercise 7. *How useful is this construction to establish the isomorphism?*

Addition, Multiplication and Exponentiation

These operations are straightforward to define by induction, following exactly the Dedekind definitions for addition and multiplication on the natural numbers. For clarity, we write α' for the successor of α.

$$\begin{aligned}
\alpha + 0 &= \alpha \\
\alpha + \beta' &= (\alpha + \beta)' \\
\alpha + \lambda &= \sup\{\, \alpha + \beta \mid \beta < \lambda \,\} \\
\alpha \cdot 0 &= 0 \\
\alpha \cdot \beta' &= \alpha \cdot \beta + \alpha \\
\alpha \cdot \lambda &= \sup\{\, \alpha \cdot \beta \mid \beta < \lambda \,\} \\
\alpha^{0} &= 0' \\
\alpha^{\beta'} &= \alpha^{\beta} \cdot \alpha \\
\alpha^{\lambda} &= \sup\{\, \alpha^{\beta} \mid \beta < \lambda \,\}
\end{aligned}$$

Properties

One can show that addition and multiplication of ordinals are both associative but they fail to be commutative.

For example, letting $1 = 0'$, $2 = 1'$, we have

$$1 + \omega = \omega \neq \omega + 1$$

and

$$2 \times \omega = \omega \neq \omega \cdot 2 = \omega + \omega$$

Exercise 8. *Show that addition and multiplication are associative.*

Exercise 9. *Show that multiplication is left-distributive but not right-distributive.*

Exercise 10. *Show that* $\alpha^{\beta+\gamma} = \alpha^{\beta} \cdot \alpha^{\gamma}$.

Justification

The definitions of ordinal arithmetic above may seem obvious, but it is worth pointing out that they faithfully represent certain natural operations on well-orders.

Suppose A and B are well-orders of order type α and β, respectively.

Let

$$C = \{ (0, a) \mid a \in A \} \cup \{ (1, b) \mid b \in B \}$$

be the disjoint union of A and B. Order C by "first A, then B". Then C is a well-order of order type of C is $\alpha + \beta$.

Likewise, setting $C = A \times B$ and ordering this set lexicographically produces a well-order of order type $\alpha \cdot \beta$.

Exercise 11. *Fill in the details for these constructions.*

Exercise 12. *Invent a similar justification for ordinal exponentiation in terms of constructing well-orders from given ones.*

Length of Induction

Ordinary induction on $\mathbb{N}$ uses a well-order of order type ω.

But the argument for the Ackermann function requires order type ω^2. Note that we can still establish the well-ordering property of this well-order using ordinary induction. For example, we could carry out the prove in Peano arithmetic.

We could also establish induction of order type ω^3 and so forth. In fact, we can get quite far within Peano arithmetic.

An ordinal α is an ε-number if $\alpha = \omega^\alpha$. Note that an ε-number is closed with respect to exponentiation.

One can show that there are lots of ε-number, in fact just as many as there ordinals. The least ε-number is usually called ε_0 (epsilon naught).

One can show that Peano arithmetic can handle inductions of length λ for any $\lambda < \varepsilon_0$ but not to ε_0 itself. In fact, induction of length ε_0 suffices to prove the consistency of Peano arithmetic.

Cardinals

Cardinals

Let's return to the issue of measuring the sizes of sets. Any well-order of order type ω requires a countably infinite carrier set.

But countable sets suffice to build well-orders of higher order types such as $\omega + \omega$, $\omega \cdot \omega$, ω^ω. Even ε_0 can easily be squeezed into a countable carrier set.

This suggests the following definition.

Definition 5. *An ordinal κ is a* cardinal *if the carrier set of any well-ordering of length κ is larger than the carrier set of any well-ordering of length $\alpha < \kappa$.*

This comes down to insisting that there is no injection

$$\{ \alpha \in On \mid \alpha < \kappa \} \to \{ \alpha \in On \mid \alpha < \beta \}$$

for any $\beta < \kappa$.

Informally, cardinalities jump when the length of a well-ordering reaches κ.

Alephs

What can we say about cardinals $Card \subseteq On$?

First off, there are the finite cardinals which consist of all ordinals less than ω – if you like, you can identify these with the natural numbers.

The first infinite cardinal is ω, though in its capacity as a cardinal rather than just plain ordinal it is often written

$$\aleph_0 \qquad \text{aleph naught}$$

This notation is, of course, due to Cantor.

But things do not end there. In fact, one can show that $Card$ is a well-ordered subclass of On, so we actually have a sequence

$$\aleph_0, \aleph_1, \ldots \aleph_\omega, \ldots \aleph_{\omega+\omega}, \ldots \aleph_{\omega^2}, \ldots \aleph_{\varepsilon_0}, \ldots, \aleph_{\aleph_1}, \ldots$$

If you find this vertigo-inducing you are quite right.

Cantor's Number Classes

This never-ending stream of alephs is important in axiomatic set-theory, but for us only the first few items are relevant.

The natural numbers aka finite ordinals form the *first number class*. The countable ordinals form the *second number class*.

Thus $\aleph_1$ is the least uncountable ordinal, the least ordinal that does not belong to the second number class.

One can show that

$$\aleph_1 \leq |\mathbb{R}|$$

but equality, Cantor's famous Continuum Hypothesis, can not be settled in the framework of standard set theory. So it is safe to assume that $\aleph_1 = |\mathbb{R}|$ or that $\aleph_1 < |\mathbb{R}|$. Unlike with the Axiom of Choice, neither option seems to be particularly natural or consequential, so the Continuum Hypothesis has not been adopted as a standard axiom.

Cardinal Arithmetic

One should note the arithmetic of cardinals is different from the arithmetic of ordinals. We won't give a detailed definition of the arithmetic operations; they are supposed to represent the effect on cardinality of disjoint union, Cartesian product and function space formation. For example, $\aleph_0 + \aleph_0 = \aleph_0$: it is easy to construct a bijection between $\mathbb{N}$ and two disjoint copies of $\mathbb{N}$.

One can show that addition and multiplication of cardinals are associative, commutative operations. Moreover

Lemma 2. *Let λ and κ be two cardinals, at least one of them infinite and neither 0. Then*

$$\lambda + \kappa = \lambda \cdot \kappa = \max(\lambda, \kappa).$$

Note that exponentiation is not mentioned here; in fact $2^\kappa > \kappa$ for all cardinals κ as Cantor's diagonal argument shows.

Inequalities

One can also derive some basic properties of inequalities between cardinalities.

Lemma 3. *Let* $\lambda \leq \kappa$ *and* $\lambda' \leq \kappa'$*. Then*

$$\lambda + \lambda' \leq \kappa + \kappa' \text{ and } \lambda \cdot \lambda' \leq \kappa \cdot \kappa'$$

Unlesss $\lambda = \lambda' = \kappa = 0 < \kappa'$ *we also have*

$$\lambda^{\lambda'} \leq \kappa^{\kappa'}.$$

There are also infinitary version of the arithmetic operations. Here is one famous result.

Lemma 4. *König*
Let $\lambda_i < \kappa_i$ *for all* $i \in I$*. Then*

$$\prod_i \lambda_i < \sum_i \kappa_i.$$

Cofinality

One interesting property of the second number class is that all its elements can be approximated by sequences of length ω.

Definition 6. *Define the cofinality of a limit ordinal α to be the length of a shortest sequence that is cofinal in $\{ \beta \in On \mid \beta < \alpha \}$. Ordinal α is regular if it has cofinality α.*

In symbols: $\mathrm{cof}\,\alpha$.

Note that $\omega + \omega$, ω^2 and the like all have cofinality ω.

Lemma 5. *All limit ordinaly in the second number class have cofinality ω. But $\aleph_1$ is regular.*

Implementing Ordinals

So far we have carefully avoided explaining how to represent ordinals and cardinals as sets and instead used concepts such as "stage of construction", "well-ordering" and so on. Here is one way to represent ordinals as ordinary pure sets due to von Neumann. All the assertions we have made above can then be proven in set-theory.

We want to define a set N_α for each ordinal α that represents α in some natural way.

First off, what is meant by natural here is that the structure

$$\langle N_\alpha, \in \rangle$$

ought to be a well-ordering of order type α.

So, for any well-ordering $\langle A, < \rangle$ there is an isomorphic order of the form $\langle N_\alpha, \in \rangle$, we don't need any order relation more complicated than $\in$.

Transitivity

If we use $\in$ as the underlying order we must have the following (order relations are transitive):

$$z \in y \in x \text{ implies } z \in x.$$

This property warrants a definition of its own.

Definition 7. *A set x is transitive if $z \in y \in x$ implies $z \in x$.*

Thus, x is transitive iff $\bigcup x \subseteq x$ iff $x \subseteq \text{pow}(x)$.

Structures that can be represented by transitive sets together with $\in$ play an important role in set theory, but we won't pursue the issue here.

Transitive Closure

As an aside, we note that for any set x we can construct the least transitive superset y, by induction on $\in$.

Definition 8. *The transitive closure of x is defined to be*

$$\mathrm{TC}(x) = \bigcap \{\, y \supseteq x \mid y \text{ transitive} \,\}.$$

We can define the transitive closure operator in by recursion along $\in$:

$$\mathrm{TC}(\emptyset) = \emptyset$$

$$\mathrm{TC}(x) = x \cup \bigcup_{z \in x} \mathrm{TC}(z)$$

Exercise 13. *Show that $y \subseteq x$ implies $\mathrm{TC}(y) \subseteq \mathrm{TC}(x)$.*

Exercise 14. *Show that the recursive definition works as advertised.*

von Neumann Ordinals

So, what should the von Neumann ordinals look like?

Needless to say, we start with $N_0 = \emptyset$.

The successor function on sets is defined by

$$S(x) = x \cup \{x\}$$

and $S(N_\alpha)$ is the successor of N_α.

So the question is what to do with limit ordinals. The answer is fairly simple: take the union of all earlier von Neumann ordinals. So

$$N_\lambda = \bigcup_{\alpha<\lambda} N_\alpha$$

Exercise 15. *Show that the von Neumann ordinals are in fact well-ordered by* $\in$.

Transitivity and Ordinals

Not every transitive set represents an ordinal, but there is a nice way to characterize these sets.

Lemma 6. *The von Neumann ordinals are precisely the transitive sets that are well-ordered by the element relationship* $\in$*.*

Proof.

An easy induction shows that all von Neumann ordinals are well-ordered by $\in$.

For the opposite direction use induction on α to show that the element in $\langle A, \in \rangle$ of rank α must be N_α (transitivity is crucial).

□

Exercise 16. *Show that every element of a von Neumann ordinal is a von Neumann ordinal using as definition "transitive and* $\in$*-well-ordered".*

Implementation versus Definition

The last lemma is rather neat; to represent well-orders in set theory we can dispense with specific order relations and just use ε. The only thing that changes is the carrier set, which has rather nice properties itself.

We note that some authors define ordinals in terms of their set-theoretic implementation. While formally correct, this approach is rather dubious since it obcures the intended properties of ordinals – they all have to be tediously discovered after the fact. Some would also gripe that this method over-emphasizes set theory.

At any rate, the crucial idea is to generalize inductive proofs and/or definitions to well-orderings other than just the natural numbers. Ordinals represent the stages in these arguments and constructions.

Summary

- Ordinals capture the notion of stages in an inductive process, including the transfinite case.
- Alternatively, we can think of them as the order types of all well-orderings.
- Ordinals can be implemented in set theory as transitive sets that are well-ordered by ε.
- Using transitive induction, one can define ordinal arithmetic which corresponds naturally to operations on the well-orderings.
- Cardinals are special types of ordinals, and carry their own arithmetic.