CDM

Number Theory I

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Comment 1

So far, we have dealt with various aspects of elementary combinatorics:

- induction
- counting
- graphs
- (finite) probability
- (finite) games

In the next few weeks we deal with algebra, then with computability.

Comment 2

You know everything in this first section.

Instead of falling asleep, think about how you might actually prove all these assertions.

In this case, a real proof should be based on Peano arithmetic: take the basic laws of addition and multiplication, plus a suitable form of induction. Figure out how to formalize induction on the integers.

Comment 3

Next week you will see a more elegant and mathematically more useful way to explain what is going on: actual algebra.

Alas, high levels of abstraction are utterly useless without a solid grounding in actual examples. This week we provide some concrete examples, next week we talk about the corresponding abstractions:

- groups
- rings and fields

Needless to say, this is also the way the area developed historically. Understanding modern math without history is very difficult.

Total Recall: Divisibility

For $a, b \in \mathbb{Z}$, a divides b iff $\exists\, c \in \mathbb{Z}\, (a \cdot c = b)$.

This is usually written $a \mid b$.

Proposition

Note that $1, -1 \mid a$ *and* $a \mid 0$. *Divisibility is reflexive, transitive and almost antisymmetric.*

Lemma (Linear Combinations)

If $d \mid a$ *and* $d \mid b$ *then* $d \mid (xa + yb)$ *for all* $x, y \in \mathbb{Z}$.

Quotients and Remainders

Theorem (Division Theorem)

Let b be positive, and a an arbitrary integer. Then there exist integers q and r such that

$$a = q \cdot b + r, \text{ where } 0 \leq r < b.$$

Moreover, the numbers q and r are uniquely determined (quotient and remainder).

In the literature this is often called the "Division Algorithm," though no algorithm is given.

Notation:

$r = a \bmod b$	remainder
$q = a \operatorname{div} b$	quotient

Primes

$p > 1$ is prime iff its only positive divisors are 1 and p.

Lemma

For every $n \geq 2$ there is a prime p such that $p \mid n$.

Theorem

There are infinitely many primes.

Lemma

If p is prime and $p \mid ab$ then $p \mid a$ or $p \mid b$.

Sample Proof

The last lemma requires a forward link to GCD, see below.

Let $d = \gcd(a, p)$.

If $d = p$ then clearly d divides a.

But otherwise $d = 1$, hence $xa + yp = 1$ for some integers x and y.

It follows that $xab + ypb = b$ and p divides b.

□

The Fundamental Theorem

Theorem

Let $n \geq 2$. Then there exist distinct primes $p_1, \ldots, p_k$ such that

$$n = p_1^{e_1} p_2^{e_2} \ldots p_k^{e_k}$$

where $e_i > 0$. The decomposition is unique up to order.

Proof.

Strong induction and the last lemma. □

But beware, finding this prime decomposition is very hard. It is exceedingly useful conceptually, but algorithmically there are issues.

GCD and Euclidean Algorithm

The greatest common divisor is defined by

$$\gcd(a, b) = \max\big(\, d \mid d \text{ divides } a, b \,\big)$$

a and b are coprime (relatively prime) iff $\gcd(a, b) = 1$.

A look at the GCD function produces an algorithm to compute it.

Lemma

- $\gcd(x, 0) = x$
- $\gcd(x, y) = \gcd(y, x)$
- $\gcd(x, y) = \gcd(y, x \bmod y)$

Example

Typical run: $a = 4234$ and $b = 4693286$.

$$\begin{aligned}
4234 &= 0 \cdot 4693286 + 4234 \\
4693286 &= 1108 \cdot 4234 + 2014 \\
4234 &= 2 \cdot 2014 + 206 \\
2014 &= 9 \cdot 206 + 160 \\
206 &= 1 \cdot 160 + 46 \\
160 &= 3 \cdot 46 + 22 \\
46 &= 2 \cdot 22 + 2 \\
22 &= 11 \cdot 2 + 0
\end{aligned}$$

The table is a (clumsy) proof that $\gcd(4234, 4693286) = 2$.

Linear Combinations

The last example suggests to take a closer look at linear combinations

$$c = x \cdot a + y \cdot b$$

where $x, y \in \mathbb{Z}$.

Obviously c is divisible by $\gcd(a, b)$.

More interestingly, we could run through the equations above backwards and write $2 = \gcd(a, b)$ as a linear combination of a and b:

$$\gcd(a, b) = 2 = 205068 \cdot a - 185 \cdot b$$

EEA

Lemma (Extended Euclidean Algorithm)

There exist integers x, y such that

$$\gcd(a, b) = x \cdot a + y \cdot b.$$

Moreover, these so-called cofactors can be computed along with the GCD.

Trace of the Euclidean algorithm. Wlog $a \geq b \geq 0$.

$$r_{i-2} = q_i \cdot r_{i-1} + r_i \quad \text{where } r_0 = a, r_1 = b$$

Hence, $r_n = 0$ for some n, and $r_{n-1} = \gcd(a, b)$. Define

$$\begin{aligned} x_0 &= 1 & y_0 &= 0 \\ x_1 &= 0 & y_1 &= 1 \\ x_i &= x_{i-2} - q_i \cdot x_{i-1} & y_i &= y_{i-2} - q_i \cdot y_{i-1} \end{aligned}$$

A simple induction shows that

$$r_i = a \cdot x_i + b \cdot y_i.$$

Example

q_i	r_i	x_i	y_i
–	1233	1	0
–	1000	0	1
1	233	1	−1
4	68	−4	5
3	29	13	−16
2	10	−30	37
2	9	73	−90
1	1	−103	127
9	0	1000	−1233

We have

$$-103 \cdot 1233 + 127 \cdot 1000 = 1 = \gcd(1233, 1000)$$

Diophantine Equation

We can also think of

$$a \cdot x + b \cdot y = c$$

as an equation, we want solutions for x and y.

Again, we clearly need $d = \gcd(a, b) \mid c$ for any solution to exist.

We can divide by the GCD and use the extended Euclidean algorithm as before.

But note that the solution is not unique: for any solution (x_0, y_0) we get infinitely many other solutions of the form

$$(x_0 + tb/d, y_0 - ta/d)$$

where $t \in \mathbb{Z}$. In fact, these are all the solutions.

Efficiency of EEA

One can implement all the necessary arithmetic in $O(k^2)$ steps for k-bit numbers. In fact addition is only $O(k)$, but for mods and remainders we need $O(k^2)$ steps.

But how often does the while-loop execute? Trivially no more than $a \geq b$ times, but that's no good at all.

Note that one must lose one bit at least at every other step. This follows from

$$r_{i-2} = q_i \cdot r_{i-1} + r_i$$

Hence total running time is $O(k^3)$ steps for k-bit inputs.

Incidentally, the worst possible input is two consecutive Fibonacci numbers. In this case, $q_i = 1$ at all times, and the algorithm just runs backwards through the Fibonacci numbers.

p-adic Valuations

Definition

Let p prime. The p-adic valuation of an integer $n \neq 0$ is the largest e such that p^e divides n, in symbols $\nu_p(n)$; we set $\nu_p(0) = \infty$.

$$\nu_p(ab) = \nu_p(a) + \nu_p(b)$$

$$a \mid b \iff \forall\, p\, (\nu_p(a) \leq \nu_p(b))$$

$$\gcd(a, b) = \prod_p p^{\min(\nu_p(a), \nu_p(b))}$$

The last formula does not yield an efficient way to compute gcd's.

The Lattice View

The natural numbers with division $\langle \mathbb{N}, | \rangle$ form a so-called lattice: a partial order where any two elements have a join (supremum) and and a meet (infimum).

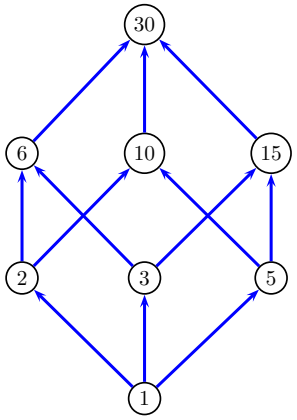
The join is the least common multiple, the meet the greatest common divisor.

If you prefer, you can think of a structure $\langle A, \sqcup, \sqcap \rangle$ where

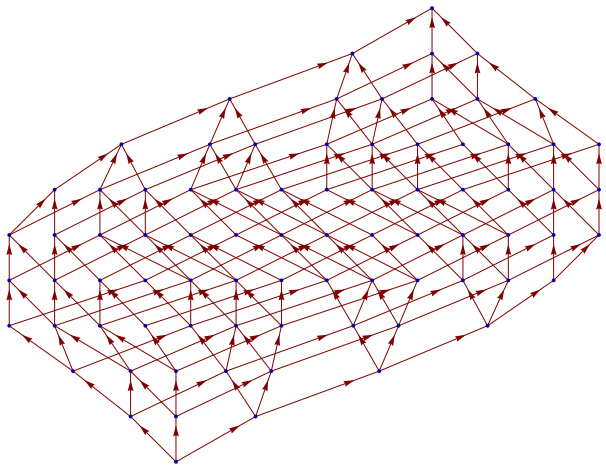
- $\sqcup$ and and $\sqcap$ are associative and commutative
- absorption holds:

$$x \sqcup (x \sqcap y) = x \qquad x \sqcap (x \sqcup y) = x$$

Divisor Lattice



148176



Exercises

Exercise

Verify that $\langle \mathbb{N}, \mathrm{lcm}, \gcd \rangle$ *really forms lattice.*

Exercise

How are lcm *and* $\gcd$ *expressed in the picture of the divisor lattice of* 30*?*

Exercise

How is the structure of prime divisors of $148176 = 2^4 3^3 7^3$ *expressed in the picture of the divisor lattice?*

Odds and Evens

Suppose we have a polynomial with integer coefficients

$$p(x) = a \cdot x^3 + b \cdot x^2 + c \cdot x + d.$$

Assume that both $p(0)$ and $p(1)$ are odd.

Claim

For all integers x*,* $p(x) \neq 0$*.*

To see why, first note that $d = p(0)$ and $a + b + c + d = p(1)$ are odd.

Even/Odd arithmetic:

$+$	*even*	*odd*
even	*even*	*odd*
odd	*odd*	*even*

$\cdot$	*even*	*odd*
even	*even*	*even*
odd	*even*	*odd*

More Odds and Evens

Hence n even (odd) implies n^k even (odd) for all $k \geq 1$.

Case 1: So for even x we get

$$p(x) = a \cdot even + b \cdot even + c \cdot even + odd = odd$$

so that in particular $p(x) \neq 0$.

Case 2: For odd x we have

$$p(x) = a \cdot odd + b \cdot odd + c \cdot odd + odd.$$

But $a + b + c$ must be even, so either 0 or 2 of these coefficients must be odd.

In both cases $p(x)$ is odd, and so not equal to 0.

Princeps Mathematicorum

Carl Friedrich Gauss, 1777-1855, my academic grand[7]-father.

Modular Arithmetic, Courtesy C. F. Gauss

Recall from equivalence relations: for $m \geq 0$, x is congruent to y modulo m

$$x \; \rho \; y \iff m \text{ divides } x - y$$

is an equivalence relation.

Notation: $x = y \pmod{m}$ or $x \equiv y \pmod{m}$.

Crucial Point: we can define arithmetic on the equivalence classes to get a structure $\mathbb{Z}_m$ as opposed to $\mathbb{Z}$:

$$[x] + [y] = [x + y]$$

$$[x] \cdot [y] = [x \cdot y]$$

We obtain modular numbers.

Representatives and Notation

A notation like $[x]$ or $[x]_m$ or $[x]_{\mathbb{Z}_m}$ for modular numbers is perfectly correct but fatally clumsy. Likewise for $+_m$ or $+_{\mathbb{Z}_m}$.

To keep notation simple, we will usually ignore the brackets and write x instead of $[x]$. And we write $+$ and $\cdot$ for addition and multiplication of modular numbers.

So, we write $\mathbb{Z}_m = \{0, 1, \ldots, m-1\}$ and sometimes think of $\mathbb{Z}_m \subseteq \mathbb{Z}$.

This is the canonical way of choosing representatives, but note that there are other possibilities. For example, for $m = 2k + 1$ we could also use

$$-k, -k+1, \ldots, -1, 0, 1, \ldots, k-1, k$$

Cayley Tables

Example ($\mathbb{Z}_2$)

$+$	0	1
0	0	1
1	1	0

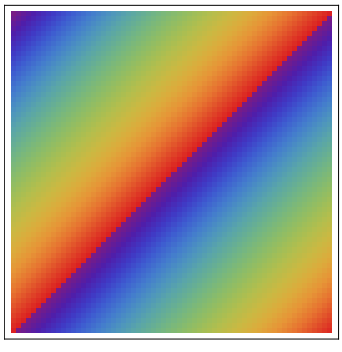
$\cdot$	0	1
0	0	0
1	0	1

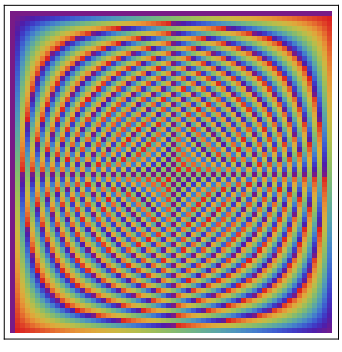
Example ($\mathbb{Z}_5$)

$+$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$\cdot$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Cayley Tables





Equivalence versus Arithmetic

There is a very important idea hiding here: equivalence relations that are compatible with arithmetic (here: on modular numbers).

Suppose ρ is an equivalence relation on $\mathbb{Z}$. ρ is a congruence iff

$$x = x' \pmod m \quad \text{and} \quad y = y' \pmod m$$

implies

$$x + y = x' + y' \pmod m$$
$$x \cdot y = x' \cdot y' \pmod m$$

This is a huge restriction compared to arbitrary equivalence relations. There are uncountably many equivalence relations on $\mathbb{Z}$, but we know all the congruences: the Gaussian relations mod m.

Exercise

Prove that any congruence on the integers is already of the form $\pmod m$.

Consequences

Think of $\bmod m$ as a function from $\mathbb{Z}$ to $\mathbb{Z}_m \subseteq \mathbb{Z}$.

Then we have

$$(x + y) \bmod m = ((x \bmod m) + (y \bmod m)) \bmod m$$
$$(x \cdot y) \bmod m = ((x \bmod m) \cdot (y \bmod m)) \bmod m$$

Note that the double application of $\bmod m$ on the right is clumsy, we'll see a better way in a while (homomorphisms).

Inevitable Clock Problem

A clock (which functions accurately) shows the hour hand positioned at a minute mark, and the the minute hand two marks away. What time is it?

Really have 60 possible positions. Equations:

$$m = h \pm 2 \pmod{60}$$
$$m = 12h \pmod{60}$$

By exploiting the congruence properties it follows that

$$11h = \pm 2 \pmod{60}$$

multiply by 11:

$$h = \pm 22 \pmod{60}$$

It's $4{:}24$ or $7{:}36$.

The Real Problem

The real question is: how does one solve equations modulo m?

Since there are only finitely many modular numbers one could, in principle, use brute force. Alas, for large moduli this is not a realistic option. (Though we will admit that sometimes brute force works modulo m).

Unfortunately life becomes fairly difficult even for quadratic equations, but we can handle linear ones $ax = c \pmod m$.

Simplifying Equations

Proposition

Let $ab = ac \pmod m$ *and* $m' = m/\gcd(a, m)$.
Then $b = c \pmod{m'}$.

In particular when a and m are coprime we can simply drop the a.

Exercise

Use p-adic valuations to prove the proposition.

Inhomogeneous Equations

First an important special case.

Lemma

The equation

$$a \cdot x = 1 \pmod m$$

has a solution if, and only if, a and m are coprime. The solution is unique modulo m, if it exists.

Proof.

A solution means that $ax - 1 = qm$, so a and m must be coprime.

In the opposite direction use the extended Euclidean algorithm to compute cofactors $ax + my = 1$.

□

Multiplicative Inverses

The situation in the lemma is very important.

The solution x such that $ax = 1 \pmod m$ is called the multiplicative inverse of a (modulo m).

Notation: $a^{-1} \pmod m$.

Example

$m = 11$.

x	1	2	3	4	5	6	7	8	9	10
x^{-1}	1	6	4	3	9	2	8	7	5	10

Note that $10 = 10^{-1}$ (no surprise, really: $10 = -1$).

So $1/2 = 6 \pmod{11}$.

Euler's Totient Function

The collection of all modular numbers that have a multiplicative inverse is usually written $\mathbb{Z}_m^*$ and called the multiplicative subgroup (see next week).

Definition

$$\mathbb{Z}_m^* = \{\, a \in \mathbb{Z}_m \mid \gcd(a, m) = 1 \,\}$$

Definition (Euler's Totient Function)

The cardinality of $\mathbb{Z}_m^*$ is written $\varphi(m)$.

Some Values

Here are the first few values of φ

$$1, 1, 2, 2, 4, 2, 6, 4, 6, 4, 10, 4, 12, 6, 8, 8, 16, 6, 18, 8, 12, 10, 22, 8, 20, 12, 18, 12, 28, 8, 30, 16, 20, 16, 24, 12, 36, 18, 24, 16, 40, 12, 42, 20, 24, 22, 46, 16, 42, 20, 32, 24, 52$$

Looks complicated.

Note that we can compute $\varphi(n)$ if we know the prime factorization of n:

- For p prime $\varphi(p) = p - 1$ and $\varphi(p^k) = (p-1)p^{k-1}$.
- For m and n coprime, $\varphi(mn) = \varphi(m)\varphi(n)$.

We will see an elegant proof of the second claim later.

Hence for prime decomposition

$$n = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$$

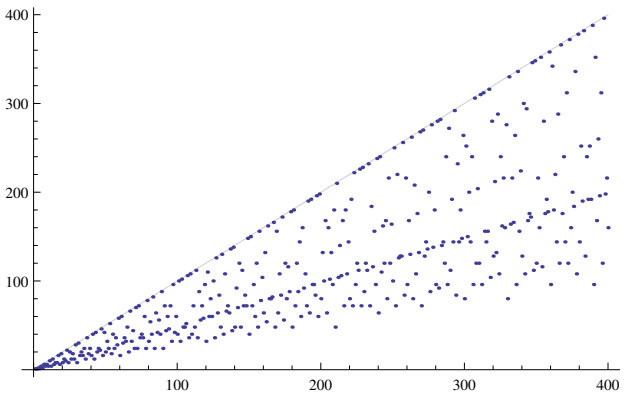
we have

$$\varphi(n) = (p_1 - 1)p_1^{e_1 - 1}(p_2 - 1)p_2^{e_2 - 1} \dots (p_k - 1)p_k^{e_k - 1}$$

Note that it is not clear how to compute $\varphi(n)$ without the prime decomposition.

In fact, the two problems are computationally closely related. Some cryptographic schemes depend on the totient function being hard to compute.

Plot



Inhomogeneous Equations, II

Lemma

In the general case

$$a \cdot x = c \pmod m$$

we have a solution if, and only if, $\gcd(a, m)$ *divides* c.

Moreover, the number of solutions is $\gcd(a, m)$.

Exercise

Prove the general case.

Recall from Iteration

Consider the additive function

$$\alpha : \mathbb{Z}_m \to \mathbb{Z}_m$$
$$x \mapsto x + s \bmod m$$

Clearly α is injective, so the orbits are all periodic (plain cycles).

Moreover, since $\alpha(x) + y = \alpha(x + y) \pmod m$ all the cycles are just rotations of each other and it suffices to understand $\mathrm{orb}(0, \alpha)$.

So we need the least $k > 0$ such that $ks = 0 \pmod m$.

Let $d = \gcd(s, m)$. Then clearly $k = m/d$.

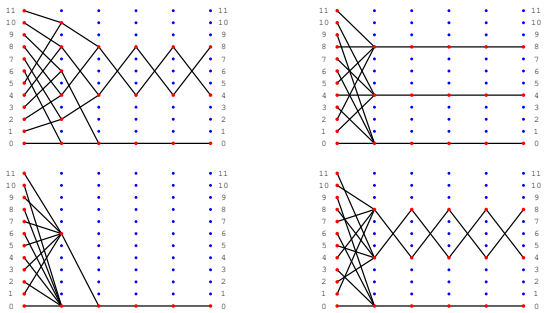
Proposition

α has $\gcd(s, m)$ distinct orbits, each of length $m/\gcd(s, m)$.

Multiplication

We also saw that the corresponding problem for multiplication is quite a bit harder.

$$\mu(x) = s \cdot x \bmod m$$



Coprime Case

Naturally one should try to understand the case when μ is injective first, since all orbits are just cycles.

Clearly μ is injective iff s and m are coprime, i.e., $s \in \mathbb{Z}_m^*$.

0 is a fixed point, so one should consider the orbit of 1 next.

So this time we need the least $k > 0$ such that $s^k = 1 \pmod m$.

This number k has a natural algebraic interpretation in $\mathbb{Z}_m^*$ that we will discuss shortly.

Mixed Case

Exercise

Determine the structure of μ orbits in the "anti-coprime case": every prime factor of m also divides s.

Exercise

Determine the structure of μ orbits when $m = pq$, p, q prime and $s = p$.

Exercise

Determine the structure of μ orbits in the general mixed case.

Modular Arithmetic and Primes

When p is prime the structure of $\mathbb{Z}_p^*$ is particularly simple:

$$\mathbb{Z}_p^* = \{1, 2, \ldots, p-1\}$$

As a consequence, we can solve all equations $ax = b \pmod p$ as long as $a \neq 0 \pmod p$.

Here are some classical results concerning prime moduli.

Wilson's Theorem

Theorem (Wilson's Theorem)

p is prime if, and only if, $(p-1)! \equiv -1 \pmod p$.

Proof.

First assume p is prime, wlog $p > 2$. We can pair off $a \in \mathbb{Z}_p^*$ and $a^{-1} \in \mathbb{Z}_p^*$. a and a^{-1} are always distinct except in the case $a = \pm 1$: the quadratic equation $x^2 = 1 \pmod p$ has at most two solutions since $x^2 - 1 = (x+1)(x-1)$.

For the opposite direction assume p fails to be prime, say, $ab = p$ for $1 < a < b < p$. But then $(p-1)!$ and p are not coprime whereas -1 and p are coprime, contradiction.

□

Little Fermat

Theorem (Fermat's Little Theorem)

If p is prime and coprime to a, then $a^{p-1} = 1 \pmod{p}$.

Proof.

Consider the map $f : \mathbb{Z}_p^* \to \mathbb{Z}_p^*$, $f(x) = ax$.

f is a bijection, so

$$a^{p-1} \prod_{x \in \mathbb{Z}_p^*} x = \prod_{x \in \mathbb{Z}_p^*} ax = \prod_{x \in \mathbb{Z}_p^*} f(x) = \prod_{x \in \mathbb{Z}_p^*} x \pmod{p}$$

Since $\varphi(p) = p - 1$, done.

□

We will see a stronger version of this in our discussion of groups.

Multiple Linear Equations

How about multiple equations with several moduli:

$$a_i x = b_i \pmod{m_i} \qquad \text{where } i = 1, \ldots, n$$

We can simplify this system a little: for a solution to exist we need that $\gcd(a_i, m_i)$ divides b_i.

So we get an equivalent equation $a_i' x = b_i' \pmod{m_i'}$ where a_i' and m_i' are coprime.

But that is equivalent to $x = c_i \pmod{m_i'}$ for some appropriate c_i.

Multiple Linear Equations, II

So let's only consider

$$x = a_i \pmod{m_i} \qquad i = 1, \ldots, n$$

Tricky in general, but for coprime moduli easy. We only consider $n = 2$.

Let $m = m_1 m_2$ and define the function

$$f : \mathbb{Z}_m \longrightarrow \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2}$$
$$f(x) = (x \bmod m_1, x \bmod m_2)$$

Claim

f is injective and hence bijective.

Proof

Proof. To see this, suppose $f(x) = f(x')$, where $0 \le x \le x' < m$.

Then

$$x' - x = q_1 m_1 = q_2 m_2.$$

But m_1 and m_2 are coprime, so $m \mid x' - x$ and therefore $x = x'$.

Since domain and codomain of f both have cardinality m, f must be a bijection by General Abstract Nonsense.

□

Hence we can solve $x = a \pmod{m_1}$ and $x = b \pmod{m_2}$: let

$$x = f^{-1}(a, b)$$

Great. But how do we find the x computationally?

CRT Example

Let $m_1 = 3$ and $m_2 = 5$, so $m = 15$.

Here is the canonical map $f : \mathbb{Z}_{15} \to \mathbb{Z}_3 \times \mathbb{Z}_5$, $f(x) = (x \bmod 3, x \bmod 5)$.

0	$(0,0)$	8	$(2,3)$
1	$(1,1)$	9	$(0,4)$
2	$(2,2)$	10	$(1,0)$
3	$(0,3)$	11	$(2,1)$
4	$(1,4)$	12	$(0,2)$
5	$(2,0)$	13	$(1,3)$
6	$(0,1)$	14	$(2,4)$
7	$(1,2)$		

By table lookup, the solution to $x = 2 \bmod 3$, $x = 1 \bmod 5$ is $x = f^{-1}(2,1) = 11$.

EEA to the Rescue

A better method is to use the EEA. Compute the cofactors:

$$\alpha m_1 + \beta m_2 = 1$$

Then

$$f(\alpha m_1) = (0, 1)$$
$$f(\beta m_2) = (1, 0)$$

whence

$$f(b\alpha m_1 + a\beta m_2) = (a, b)$$

So the solution is $x = b \cdot \alpha m_1 + a \cdot \beta m_2$.

CRT Example II

As we have seen, the solution to

$$x = 2 \bmod 3 \qquad x = 1 \bmod 5$$

is $x = 11$.

Here is the computationally superior solution: determine cofactors

$$(-3) \cdot 3 + 2 \cdot 5 = 1$$

which produce a solution

$$x = 1 \cdot (-3) \cdot 3 + 2 \cdot 2 \cdot 5 = 11$$

Chinese Remainder Theorem

Our result also holds for more than 2 equations (and is very old).

Theorem (CRT)

Let m_i, $i = 1, \ldots, n$ be pairwise coprime. Then the equations

$$x = a_i \pmod{m_i} \qquad i = 1, \ldots, n$$

have a unique solution in $\mathbb{Z}_m$, $m = m_1 m_2 \ldots m_n$.

This follows from repeated application of the solution for $n = 2$ since m_1 and $m_2 \ldots m_2$ are also coprime.

Computation

How do we compute the solution for $n > 2$? We could use the method for $n = 2$ recursively, but that is a bit tedious. Here is a better way.

Define

$$c_i = m/m_i$$

so that $c_i = 0 \pmod{m_j}$, $i \neq j$, but c_i and m_i are coprime. Use EEA to find inverses

$$\alpha_i c_i = 1 \pmod{m_i}$$

Then

$$x = a_1\alpha_1 c_1 + a_2\alpha_2 c_2 + \ldots a_n\alpha_n c_n \pmod{m}$$

The Bored Bank Clerk

> A bored bank clerk has a big pile of one-dollar bills in front of him. He rearranges the bills first in groups of 2, then 3, and 4, 5, 6, 7, 8, 9, 10 and 11. In all cases except the last, one bill is left over. In the last case, no bill is left over.
>
> How big is the original pile?

Note that we cannot use the CRT directly: the moduli are not coprime.

But $x = 1 \bmod 8$ implies $x = 1 \bmod 4$ and $x = 1 \bmod 2$.

Moreover, $x = 1 \bmod 9$ implies $x = 1 \bmod 3$. And both together imply $x = 1 \bmod 6$.

Similarly we can drop the condition modulo 10.

So, the whole system boils down to

$$x = 1 \pmod{m} \text{ where } m = 5, 7, 8, 9$$
$$x = 0 \pmod{11}$$

We have $m = 27720$ and

$$(4, 3, 1, 5, 1) = (5544, 3960, 3465, 3080, 2520)^{-1} \pmod{5, 7, 8, 9, 11}$$

and thus

$$x = 4 \cdot 5544 + 3 \cdot 3960 + 1 \cdot 3465 + 5 \cdot 3080$$
$$= 25201 \bmod 27720$$

A Generalization

In general, a solution may exist even if some of the moduli are not coprime. This is expressed in the following generalization.

Theorem (Generalized CRT)

The equations

$$x = a_i \pmod{m_i} \qquad i = 1, \ldots, n$$

have a solution if, and only if, for all $i \neq j$*:*

$$a_i = a_j \pmod{\gcd(m_i, m_j)}$$

The solution is unique modulo $m = \operatorname{lcm}(m_1, m_2, \ldots, m_n)$.

A Closer Look

In the CRT example from above we had

$$11 + 8 = 4 \bmod 15 \qquad (2,1) + (2,3) = (1,4) \bmod (3,5)$$
$$11 \cdot 8 = 13 \bmod 15 \qquad (2,1) \cdot (2,3) = (1,3) \bmod (3,5)$$

It looks like

$$f(x + y) = f(x) + f(y)$$
$$f(x \cdot y) = f(x) \cdot f(y)$$

Here we are joyfully abusing notation, we do not distinguish between the $+$ operation in $\mathbb{Z}_{15}$ and its counterpart in $\mathbb{Z}_3 \times \mathbb{Z}_5$.

Anal-Retentive Notation

We could write something like

$$f(x +_{15} y) = f(x) +_{3,5} f(y)$$

or even

$$f(x +_{\mathbb{Z}_{15}} y) = f(x) +_{\mathbb{Z}_3 \times \mathbb{Z}_5} f(y)$$

but that's awful to look at.

It's just operator overloading, no problem for a CS major.

Logarithms

Here is the classic historical example of such a map:

$$\log : \mathbb{R}^+ \to \mathbb{R}$$

which translates multiplication into addition (next week: a group isomorphism from $\langle \mathbb{R}^+, \cdot, 1 \rangle$ to $\langle \mathbb{R}, +, 0 \rangle$).

We can compute products of (positive) reals by

$$x \cdot y = e^{\log x + \log y}$$

Makes a huge difference: $O(k)$ plus table lookup rather than $O(k^2)$ where k is the number of decimal digits.

Of course, we have to compute a logarithm table first–but only once. It took John Napier some 20 years to construct such a table in the early 1600s.