CDM

Combinatorics

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1 Counting

- Multinomials
- Inclusion/Exclusion

Counting is perhaps the most fundamental activity in mathematics.

How many

- poker hands with 3 aces
- 00-free binary lists of length k
- binary trees on k nodes
- prime numbers
- rational numbers
- real numbers
- C programs

are there?

The first three questions seem reasonable.

We would expect an answer like “12345”, or $k(k-1)$ or some such.

But for the rest the intuitive answer is simply “infinitely many”. To make sense out of this, we have to explain more carefully what we mean by “infinite”. It turns out that there are levels on infinity, and one can have a classification rather similar to the finite case.

Actually, infinite counting is often a whole lot easier.

But let’s postpone this for a moment.

Everybody knows how to count, but let's be a bit formal about this.

By counting we mean determining the cardinality of some set S.

As long as S is finite (first three examples), this means to find the right number n and to enumerate the set as

$$S = \{a_1, \ldots, a_n\}.$$

In other words, we have to establish a bijection $f : [n] \to S$ as in the next table:

$$\begin{array}{cccccc} 1 & 2 & 3 & \ldots & n-1 & n \\ \updownarrow & \updownarrow & \updownarrow & \ldots & \updownarrow & \updownarrow \\ a_1 & a_2 & a_3 & \ldots & a_{n-1} & a_n \end{array}$$

Usually one is only interested in n, but sometimes one needs to find an actual bijection

$$f : [n] \to S$$

There are lots of possible bijections ($n!$ to be precise).

Find one that is easy to compute and places the elements into some natural order.

We also want $f^{-1} : S \to [n]$ to be easily computable.

These bijections are called ranking (f^{-1}) and unranking functions (f).

We know the cardinality of $S = \mathfrak{P}([n])$ is 2^n.

To get a bijection $f : [2^n] \to \mathfrak{P}([n])$ we can use binary expansions.

$$x = 1 + \sum_{i<n} x_i \cdot 2^i$$
$$f(x) = \{\, i+1 \mid x_i = 1 \,\}$$

This is just the old trick of thinking of the binary expansion (padded to n digits) as a bitvector.

Always think about these bijections in the following.

One would like a simple answer, using only basic arithmetic: sums, products, exponentials, factorials, logarithms, plus perhaps a little more.

We want a closed form solution, not some recurrence (though finding a recurrence may be an important step).

As it turns out, we often need some special functions such Fibonacci numbers, binomials, harmonic numbers, Stirling numbers, ...

To find nice solutions it is helpful to have a library of combinatorial identities: equations that reduce once counting problem to another.

Example

The number of bijections on $[n]$ is $n!$.

The number of functions from $[m]$ to $[n]$ is n^m.

In CS, we often have to count words, sets, lists, leaves, binary trees, graphs, recursive calls, and so on.

This is the subject of **combinatorics**.

Vast field, lots of amazing tricks, but you only have to know a few basic facts to attack a fairly large number of problems.

So how about the poker hands with 3 aces?

There are 4 aces total, so we need to figure out how many ways we can select 3 of them.

Then we have to pick 2 of the remaining $52 - 4 = 48$ cards, and multiply the two numbers together.

Important method: break up into subproblems. Here have two subproblems of the same type. Let's do this systematically.

Here are the two most basic counting rules.

The Sum Rule

If one event can occur in n ways, and another in m ways, then the two events can occur in $n + m$ ways (one or the other, not both).

The Product Rule

If one event can occur in n ways, and another in m ways, then the two events together can occur in $n \cdot m$ ways.

OK, this is somewhat embarrassing.

But, we have already used the Product Rule in the poker problem: number of ways to get the aces times number of ways to get the rest.

What these rules really mean is this.

Let A and B be finite sets:

- If $A \cap B = \emptyset$ then $|A \cup B| = |A| + |B|$.
- $|A \times B| = |A| \cdot |B|$.

Sometimes it is easier to think about "events" than about the cardinalities of sets, that's all.

However, one should not underestimate the importance of the right psychological setup. Some apparently hard problems melt away once the right approach has been found.

An application of the product rule (plus induction).

Claim

There are n^m functions from $[m]$ to $[n]$.

Note that $f(i)$ is independent of $f(j)$ for any $j \neq i$. So, we can pick $f(1)$, $f(2)$, ..., $f(m)$ in n^m ways.

Here is an alternative approach.

Claim

$$|\underbrace{[n] \times [n] \times \ldots \times [n]}_{m}| = n^m$$

Alternatively, we can identify $f : [m] \to [n]$ with the m-tuple $(f(1), \ldots, f(m))$ of the function values. So $[m] \to [n]$ is the same as $[n] \times [n] \times \ldots \times [n]$.

There are several other counting problems in connection with functions.

How many

- functions
- injective functions
- surjective functions
- bijective functions
- strictly increasing functions
- nondecreasing functions

from $[m]$ to $[n]$ are there?

Think about a these functions as arrays.

Last question: How many sorted arrays of size m with entries in $[n]$ are there?

The first argument we used to count functions is very important: “repeatedly select something out of a group of objects”.

You are standing in front of a box containing n balls.

How many ways are there to pick k balls,
one at a time, from the box?

This is often called an urn model.

Urn sounds like graveyard, so let’s call it a box instead.

May sound clear and completely specified, but there are several subtle variants.

Suppose the box contains elements $\{a, b, c, d, e\}$.

- Order does not count: combinations
  Selection a, b, c is considered the same as c, a, b.
- Order does count: permutations
  Selection a, b, c is considered different from c, a, b.
- With replacement:
  Can select a, a, a, b.
- Without replacement:
  Cannot select any object twice.

So combinations without replacement correspond to subsets, but with replacement we get multi-sets.

Applications

How about the poker problem?

Does order count?

Do we have replacement?

How about counting functions $[m] \to [n]$?

Does order count?

Do we have replacement?

How about counting injective functions?

Does order count?

Do we have replacement?

Boxes and Balls versus Formal Models

Boxes and balls are very helpful to develop intuition, but it is also important to be able to pin down the real mathematical content (think implementations for example).

- A combination is really a set, or a multi-set in the case of replacement.
  Multi-set means: there may by multiple occurrences, but order does not count.
- A permutation is really a sequence, a function with domain $[k]$.
  Without replacement we have an injective function, with replacement an arbitrary one.

So, we are really counting sets, multi-set, functions and injective functions.

For the number of k-combinations and k-permutations of n objects write

$$\begin{aligned} C(n,k) &= \text{ no. of } k\text{-combinations of } n \text{ objects} \\ C_r(n,k) &= \text{ same with replacement} \\ P(n,k) &= \text{ no. of } k\text{-permutations of } n \text{ objects} \\ P_r(n,k) &= \text{ same with replacement} \end{aligned}$$

We already know $P_r(n,m) = n^m$:
the number of m-tuples of elements of $[n]$,
the number of functions $[m] \to [n]$.

We also know $P(n,n) = n!$:
the number of permutations (bijections) of $[n]$.

But how about k-permutations?

Lemma

$$P(n,k) = \frac{n!}{(n-k)!}$$

Proof.

Start with all $n!$ permutations of $[n]$.

Then chop off the last $n-k$ items.

$$a_{\pi(1)}, a_{\pi(2)}, \ldots, a_{\pi(k)}, \underbrace{a_{\pi(k+1)}, \ldots, a_{\pi(n)}}$$

Produces every k-permutation exactly $(n-k)!$ times.

Divide by $(n-k)!$ to compensate for over-counting. $\square$

Example

There are $P(4,3) = 4!/1! = 24$ 3-permutations over $\{a, b, c, d\}$:

$$\begin{gathered}(a,b,c),(a,b,d),(a,c,b),(a,c,d),(a,d,b),(a,d,c),\\ (b,a,c),(b,a,d),(b,c,a),(b,c,d),(b,d,a),(b,d,c),\\ (c,a,b),(c,a,d),(c,b,a),(c,b,d),(c,d,a),(c,d,b),\\ (d,a,b),(d,a,c),(d,b,a),(d,b,c),(d,c,a),(d,c,b)\end{gathered}$$

Example

10-permutations over $[20]$:

$$P(20,10) = 20!/10! = \frac{2432902008176640000}{3628800} = 670442572800$$

This is quite typical: the numbers become huge very quickly. Many algorithms die miserable deaths because of this blow-up.

For factorials there is a nice approximation formula due to Stirling.

$$n! \approx \sqrt{2\pi n} \cdot (n/e)^n$$

One can get a very precise bound on the error if needed:

$$n! = \sqrt{2\pi n} \cdot (n/e)^n \cdot (1 + \frac{1}{12n} + O(n^{-2})).$$

Example

$P(20, 10) \approx 6.7046 \cdot 10^{11}$

Falling Factorial Powers

Here is some handy notation.

For any (real) number x and non negative integer k define the falling factorial power

$$x^{\underline{k}} = x(x-1)(x-2)\ldots(x-k+1).$$

So we have

$$P(n,k) = n^{\underline{k}}$$
$$P_r(n,k) = n^k$$

Example

There are $n^{\underline{m}}$ injective functions from $[m]$ to $[n]$.

Can think of $x^{\underline{k}}$ as a polynomial in x:

$$x^{\underline{5}} = x^5 - 10\,x^4 + 35\,x^3 - 50\,x^2 + 24\,x$$

What's the Next Term?

Consider the sequence (a_n) starting with

$$10, 10, 12, 16, 22, 30, ??$$

What is the next term?

Any educated guesses?

What if the sequence were

$$-1, -2, -1, 8, 31, 74, ??$$

or

$$1, 1, 0, -1, 0, 7, 28, 79, ??$$

As stated, the problem is meaningless: anything could be the next term.

But assume that $a_n = p(n)$ where $p(x)$ is some simple function.

In particular, let's say $p(x)$ is a polynomial. We could try to find coefficients c_i such that

$$p(x) = \sum_{i=0}^{k} c_i \cdot x^i$$

matches the given values for $x = 0, \ldots, 5$.

Actually, we don't even know the degree k. $k = 5$ will be enough, no matter what, but perhaps something smaller will work.

Could resort to calculus type interpolation: fit a polynomial to the data points (i, a_i) for $1 = 0, \ldots, 5$.

Let's use magic instead. Write a table

	0	1	2	3	4	5	6
0 :	10	10	12	16	22	30	
1 :	0	2	4	6	8		
2 :	2	2	2	2			
3 :	0	0	0				

We're just taking differences between consecutive terms.

Doing this 3 times seems to produce 0 everywhere.

Now we can reverse-engineer the whole sequence ...

What's Going On?

Differences and falling factorials coexist very peacefully.

Claim

$$(x+1)^{\underline{k}} - x^{\underline{k}} = k \cdot x^{\underline{k-1}}$$

But note that $(x+1)^k - x^k$ is a big mess. Let's write

$$(\Delta f)(x) = f(x+1) - f(x)$$

This looks a lot like differentiation, but there are no limits here. So

$$\Delta x^{\underline{k}} = k \cdot x^{\underline{k-1}}$$

and by iterating Δ k times we get

$$\Delta^k x^{\underline{k}} = k!$$

a constant. Hence $\Delta^{k+1} x^{\underline{k}} = 0$.

Linearity

But note that Δ is linear in the sense

$$\Delta(f+g) = \Delta f + \Delta g$$
$$\Delta(c \cdot f) = c \cdot \Delta f$$

So if we write

$$p(x) = \sum_{i=0}^{k} c_i \cdot x^{\underline{i}}$$

we have $\Delta^{k+1} p = 0$.

For the table, we have values of a degree $k-1$ polynomial in row 1. And so on: row k will be constant, and row $k+1$ all 0.

So magic works, as long as the original sequence really is polynomial.

To count combinations we introduce another useful concept: a binary choice sequence.

$$(0, 0, 1, 0, 1, \ldots, 1, 1, 0)$$

Think of making n Yes/No decisions.

- Yes: pick the ith ball in the box.
- No: don't pick the ith ball in the box.

Clearly, there are 2^n choice sequences of length n.

Claim

There are $C(n, k)$ choice sequences of length n that contain exactly k many 1's.

Some Values

Some values of $C(n, k)$ are easy to compute:

$$
\begin{aligned}
C(n, 0) &= C(n, n) = 1 \\
C(n, 1) &= n \\
C(n, 2) &= n(n-1)/2
\end{aligned}
$$

But we want a nice, general formula.

Lemma

$$C(n, k) = \frac{n!}{k!(n-k)!} = \frac{n^{\underline{k}}}{k!}$$

Note that

$$C(n, k) = C(n, n-k)$$

Proof.

We use our old trick: over-count, and then correct the result.

First assume all the 0's and 1's are distinct.

$$1_1, 1_2, \ldots, 1_k, 0_1, 0_2, \ldots, 0_{n-k}$$

Can be arranged in $n!$ ways (permutations).

But since really $0_i = 0$, and $1_i = 1$, we over-counted by $k!(n-k)!$.

Hence $C(n,k) = n!/k!(n-k)!$.

□

Distinguishing the indistinguishable is a very important idea.

How many ways can one arrange the letters in Mississippi?

Pretend

$$M_1 i_1 s_1 s_2 i_2 s_3 s_4 i_3 p_1 p_2 i_4$$

Letter counts: M: 1, i: 4, s: 4, p: 2.

$$\frac{11!}{1!\ 4!\ 4!\ 2!} = \frac{39916800}{1152} = 34650$$

We can think of a choice sequence as a bitvector (characteristic function) representing a subset of $[n]$. The number of 1's in the sequence is the cardinality of the set.

Lemma

There are $C(n,k)$ k-element subsets of an n-set.

An interesting computational problem is to generate all the k-subsets of $[n]$.

Of course, without constructing the whole power set first: that has exponential size, but there are only $C(n,k) = \Theta(n^k)$ subsets of fixed size k.

For fixed k you can use k nested loops, but how about a function `powerset(n,k)` of two variables?

At long last, we can really handle our poker problem. We're talking 5-card hands here.

There are $C(4,3) = 4$ ways to select the 3 aces.

There are $C(48,2) = 1128$ ways to select the other 2 cards.

Hence the total answer is $4 \cdot 1128 = 4512$.

And for 2 aces the answer would be

$$C(4,2) \cdot C(48,3) = 6 \cdot 17296 = 103776$$

The total number of poker hands is

$$C(52,5) = 2598960$$

How many ways to get a flush (all cards of the same suit)?

$$4 \cdot C(13, 5) = 5148$$

How many ways to have at least one card of the each suit? E.g.,
$2\clubsuit + 1\diamondsuit + 1\heartsuit + 1\spadesuit$

$$4 \cdot C(13, 2) \cdot C(13, 1)^3 = 685464$$

How many ways to have exactly three of a kind?

$$4 \cdot C(13, 3) \cdot C(48, 2) = 18048$$

Definition

The binomial coefficient $\binom{n}{k}$ is defined to be the coefficient of x^k in the expansion of $(1+x)^n$. Here $0 \le k \le n$.

Read "n choose k". Also written C^n_k.

Example

For $n = 10$, $k = 0, 1, \ldots, n$, we get

$$1, 10, 45, 120, 210, 252, 210, 120, 45, 10, 1$$

Theorem (Binomial theorem)

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}.$$

Proof. We may assume $b \neq 0$. Then

$$(a+b)^n = (1 + a/b)^n \cdot b^n$$

Done by the definition of binomial coefficient. □

This problem appears to have been tackled first by the eleventh century Persian astronomer Omar Khayyam.

From the theorem we get a proof of

$$\binom{n}{k} = \binom{n}{n-k}.$$

Do you see why?

Like Fibonacci numbers, binomials appear in many places.

Donald Knuth devotes all of chapter 5 of “Concrete Mathematics” to binomial coefficients.

A little Knuth story: in 1972 someone published a paper on an improved merge sort algorithm.

The improvement supposedly was (number of saved transfers):

$$t = \sum_{i=0}^{n} i \binom{m-i-1}{m-n-1} \Big/ \binom{m}{n}$$

The author even thanks the referee for having produced this much simplified formula – his original mess was worse.

Alas, Knuth produces another simplification:

$$t = \frac{n}{m - n + 1}$$

So: knowing a bit about binomials is crucial.
The original formula is just about useless!

Embarrassing fact: even the Computer Algebra system Mathematica can do the simplification.

Are binomials a new idea?

No, they are just another example of choice sequences: in each term $(x+1)$ in the product

$$(x+1)(x+1)(x+1)\dots(x+1)(x+1)$$

we have to pick either 1 or x.

To get x^k, we have to pick x exactly k times.

Lemma

$\binom{n}{k} = C(n,k) = \frac{n!}{k!(n-k)!}$

It follows immediately that

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n$$

Can also squeeze out information the other way around:

$$\sum_i \binom{n}{2i+1} = \sum_i \binom{n}{2i}$$

Just set $a = 1$, $b = -1$ in the binomial theorem.

By the last equation, there is a bijection

$$\mathfrak{P}_{odd}(A) \longleftrightarrow \mathfrak{P}_{even}(A)$$

But we don't know what such a bijection might look like.

Exercise

Find an explicit bijection between the even- and odd-cardinality subsets of $[n]$.

Let A be an arbitrary finite set and pick an element $a \in A$.

Define

$$f : \mathfrak{P}(A) \to \mathfrak{P}(A)$$
$$f(X) = \begin{cases} X - \{a\} & \text{if } a \in X, \\ X \cup \{a\} & \text{otherwise.} \end{cases}$$

It is clear that $f \circ f = I$.

Hence f is a bijection.

Thinking of f as a permutation of $\mathfrak{P}(A)$ we can see that its cycle decomposition contains only 2-cycles.

Each 2-cycle associates an even cardinality set with an odd cardinality set.

Approximations

We can use Stirling's approximation to get an idea of the size of $C(n, k)$.

For example, the central binomial coefficient is

$$C(2n, n) \approx \frac{1}{\sqrt{\pi n}} \cdot 2^{2n}$$

Thus, $C(100, 50) \approx 1.008913 \cdot 10^{29}$.

Since there are only 2^{2n} subsets of $[2n]$, a surprisingly large number of these subsets has size n.

- Counting

2. Multinomials

- Inclusion/Exclusion

How about the coefficients in the expansion of $(a+b+c)^n$ or $(a+b+c+d)^n$? These coefficients are called multinomial coefficients and usually written

$$C(n; k_1, k_2, \ldots, k_m) = \binom{n}{k_1, k_2, \ldots, k_m}$$

where $\sum k_i = n$.

Theorem

Multinomial theorem

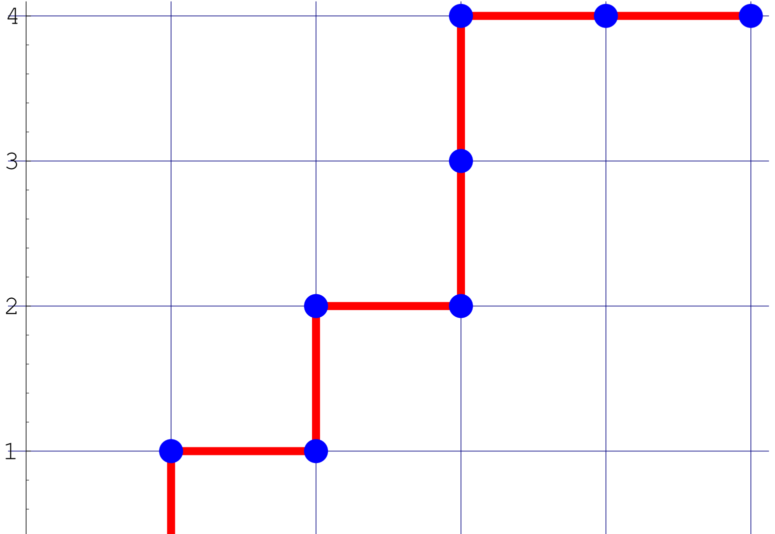
$$(x_1 + \ldots + x_m)^n = \sum \binom{n}{k_1, k_2, \ldots, k_m} x_1^{k_1} x_2^{k_2} \ldots x_m^{k_m}$$

Note that

$$\binom{n}{k_1, k_2, \ldots, k_m} = \frac{n!}{k_1! k_2! \ldots k_m!}$$

Example

One excellent model for choice sequences and binomials is to think of North-East walks in a grid.



Proof 1

There are exactly $n + m$ steps on any walk from $(0, 0)$ to (n, m).

Moreover, exactly n of these steps are "East", and m are "North".

This follows easily from the fact that any walk $(x_i, y_i)_i$ is monotonic with respect to both the x-axis and the y-axis: $i < j$ implies $x_i \leq x_j$ and $y_i \leq y_j$.

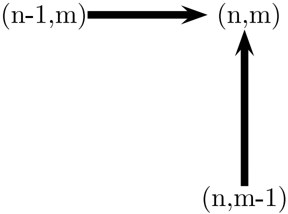
But then there are $C(n + m, n) = C(n + m, m)$ paths from $(0, 0)$ to (n, m).

Proof 2

Let's write $p(n,m)$ for the number of walks from $(0,0)$ to (n,m). Clearly

$$p(0,m) = p(n,0) = 1$$

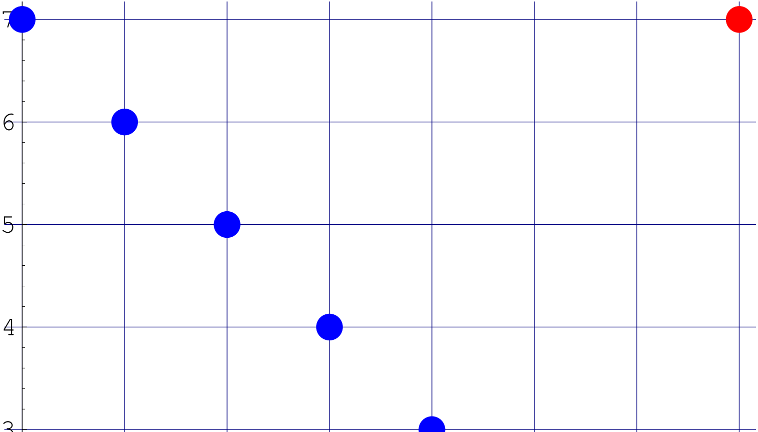
For any interior node we have

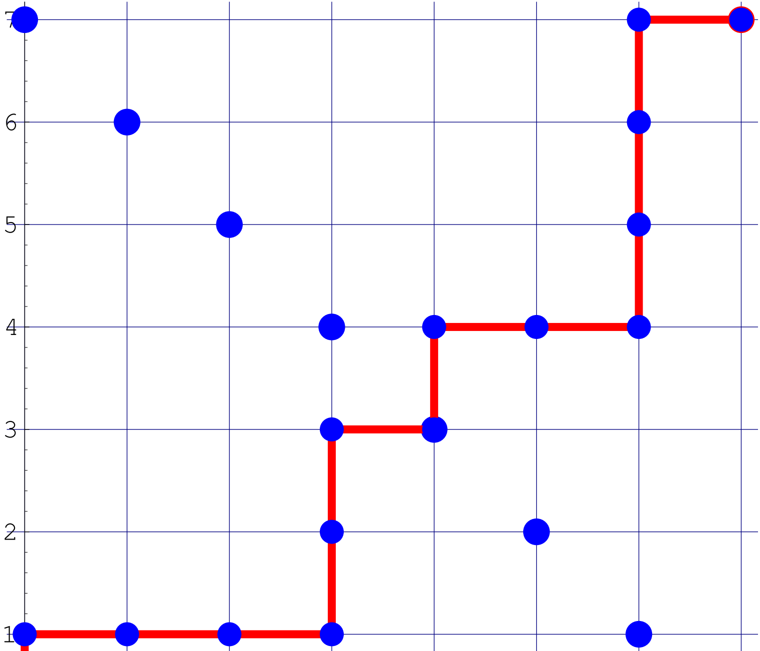


By induction, $p(n,m) = p(n-1,m) + p(n,m-1) =$

Lemma

$$\sum \binom{n}{i}^2 = \binom{2n}{n}$$




7
6
5
4
3
2
1

Lemma

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$$

$$\binom{n}{k} = \frac{n}{n-k}\binom{n-1}{k}$$

$$\binom{n}{k} = \frac{n-k+1}{k}\binom{n}{k-1}$$

Addition Rule:

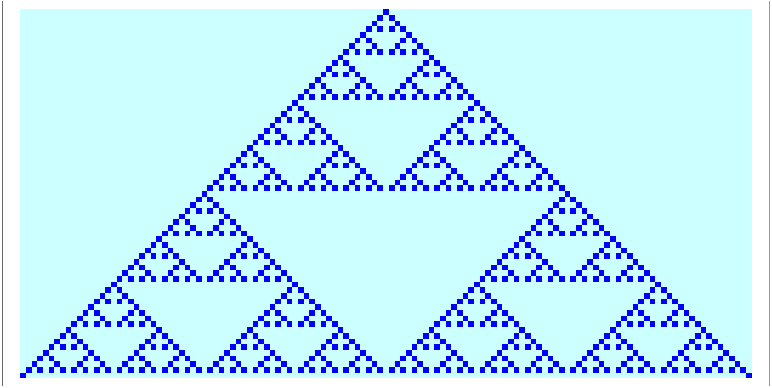
$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

The Addition Rule is the basis for Pascal's triangle . . .

```
                1
              1   1
            1   2   1
          1   3   3   1
        1   4   6   4   1
      1   5  10  10   5   1
    1   6  15  20  15   6   1
  1   7  21  35  35  21   7   1
1   8  28  56  70  56  28   8   1
```

Ponder deeply.

Same modulo 2



All the identities above have straightforward algebraic proofs using the fact that

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Just plug in the factorial expressions, simplify a bit, done.

Correct, but infinitely boring.

Much more interesting are proofs based on combinatorial meaning. E.g., for the first identity

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$$

we can argue in terms of k-subsets of an n-set.

Consider a set $A = \{a_1, \ldots, a_n\}$.

Set $A_i = A - \{a_i\}$.

Write $\mathfrak{P}_k(S)$ for all k-subsets of S.

Then

$$\mathfrak{P}_k(A) = \bigcup_{i=1}^{n} \{\, X \cup \{a_i\} \mid X \in \mathfrak{P}_{k-1}(A_i) \,\}$$

Each of the collections $\mathfrak{P}_{k-1}(A - \{a_i\})$ in the union has $\binom{n-1}{k-1}$ elements, and there are n of them.

But: we are over-counting by a factor of k: each k-element subset can be generated in exactly k ways by throwing the missing element a_i back in.

So, we have to divide by k in the end.

□

There are countless equations involving binomials that range from the obvious to the impossible-to-prove.

$$\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$$

$$\binom{n+m}{k} = \sum_i \binom{n}{i}\binom{m}{k-i}$$

Proof. For the first equation, think about pairs (A, B) where $B \subseteq A \subseteq [n]$ and $|A| = m$, $|B| = k$.

Could first pick A, and then B, or first B and then A.

Second equation is an exercise. □

Here is another important class of problems:

> How many ways are there to place
> n balls into k boxes?

Again, there are several cases:

- Objects distinguishable: think of balls numbered $1, 2, \ldots, n$.
- Objects indistinguishable: think of n identical balls.
- Boxes distinguishable: think of boxes numbered $1, 2, \ldots, k$.
- Boxes indistinguishable: think of k identical boxes.

One more twist: sometimes none of the boxes are allowed to be empty (this is much, much harder). So there are 8 possibilities, but we won't treat them systematically.

The non-empty condition pops up naturally e.g. when we try to count the number of surjective functions $[n] \to [k]$:

we need to distribute n distinguishable balls into k distinguishable boxes so that no box remains empty.

Incidentally, for $n \geq k$ the answer is

$$\sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^n.$$

A horror.

For $n = 5, k = 3$ we get 150 surjections.

Only Boxes Distinguishable

How many different ways can we distribute n indistinguishable balls into k distinguishable boxes?

Sounds like a new problem, but isn't really.

Example

8 balls and 4 boxes

$$\bullet\bullet|\bullet\bullet\bullet|\bullet|\bullet\bullet \qquad \to 2,3,1,2$$
$$\bullet\bullet|\bullet\bullet\bullet||\bullet\bullet\bullet \qquad \to 2,3,0,3$$
$$\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet||| \qquad \to 8,0,0,0$$

Every distribution of balls can be represented as a sequence of bullets and lines.

So putting n balls into k boxes boils down to placing $k-1$ vertical lines in a line of n bullets.

There are a total of $n+k-1$ possible positions for the bars.

Hence there are $\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$ ways of selecting the $k-1$ positions.

So the answer is:

$$\binom{n+k-1}{n}.$$

Example

$n = 8$, $k = 4$: 165

$n = 20$, $k = 10$: 10015005

Lemma

The number of nondecreasing functions $[p] \to [q]$ *is*

$$\binom{p+q-1}{p}$$

Example

$p = 3$ and $q = 4$: 20 nondecreasing functions.

$$(1,1,1),(1,1,2),(1,1,3),(1,1,4),(1,2,2),(1,2,3),(1,2,4),$$
$$(1,3,3),(1,3,4),(1,4,4),(2,2,2),(2,2,3),(2,2,4),(2,3,3),$$
$$(2,3,4),(2,4,4),(3,3,3),(3,3,4),(3,4,4),(4,4,4)$$

How do we reduce this problem to something already known?

A Trick

Here is a trick: Place $q-1$ indistinguishable balls into $p+1$ distinguishable boxes.

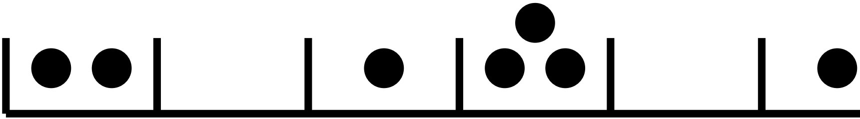
Define

$$f(1) = 1 + \text{no. balls in box } 1$$
$$f(i+1) = f(i) + \text{no. balls in box } i+1$$

Clearly, this function is nondecreasing.

Example

$p = 5$, $q = 8$.



But every nondecreasing function can be obtained in this way: the number of balls in box $i+1$ is just $f(i+1) - f(i) \geq 0$.

In other words, there is a bijection between nondecreasing functions $[p] \to [q]$ and some $p+1$-tuples of natural numbers:

$$(b_1, b_2, \ldots, b_{p+1}) \text{ where } \sum b_i = q - 1$$

corresponds to

$$f(k) = 1 + \sum_{i \leq k} b_i.$$

Box $p+1$ is just for over-flow.

Note: $\binom{p+q-1}{p}$ is thus the number of sorted arrays of length p with entries in $[q]$.

We still don't have a formula for $C_r(n, k)$.

Or do we? Here are the 20 3-combinations over $\{a, b, c, d\}$:

$$(a, a, a), (a, a, b), (a, a, c), (a, a, d), (a, b, b), (a, b, c), (a, b, d),$$
$$(a, c, c), (a, c, d), (a, d, d), (b, b, b), (b, b, c), (b, b, d), (b, c, c),$$
$$(b, c, d), (b, d, d), (c, c, c), (c, c, d), (c, d, d), (d, d, d)$$

But this is really just another way of talking about non-decreasing functions: we can sort each combination, to get such a function.

It's crucial that we are dealing with combinations here, not permutations.

Hence we have:

Lemma

$$C_r(n, k) = C(n + k - 1, k)$$

So, combinations all boil down to binomial coefficients.

$$C(n, k) = \binom{n}{k}$$
$$C_r(n, k) = \binom{n + k - 1}{k}$$

Application: Fred's Books

Fred Hacker has 10 math books, 12 physics books, and 15 CS books.

- How many ways can they be put on a bookshelf?
- What if we don't distinguish between books in each field?
- What if we want to keep books in each field contiguous?
- What if we want every math book to be followed by a physics book?
- What is the relative order of these numbers?

- 13763753091226345046315979581580902400000000
  $1.3764 \cdot 10^{43}$

- 6055322318004960
  $6.0553 \cdot 10^{15}$

- 13638005412495768944640000000
  $1.3638 \cdot 10^{28}$

- 10888869450418352160768000000
  $1.0889 \cdot 10^{28}$

Do you see where these numbers come from?

Suppose we want to count the number of ways n distinguishable balls can be placed into k indistinguishable boxes so that no box remains empty.

This appears to be quite difficult, there is no obvious reduction to any or our previous results.

Definition

Call this number the Stirling number (of the second kind), written $S_2(n, k)$.

These Stirling numbers are an important counting device. There are also Stirling numbers of the first kind, but they are not as important.

Thus, $S_2(n, k)$ is the number of ways an n-set can be partitioned into k nonempty blocks. In other words, $S_2(n, k)$ is the number of equivalence relations on $[n]$ with exactly k blocks.

Allowing Emptiness

What if we drop the non-emptiness condition? Then we get the number of all equivalence relations with at most k blocks, or

$$\sum_{i=1}^{k} S(n,i)$$

which does not look any easier to deal. An important special case is the total number of equivalence relations on $[n]$, the so-called Bell number

$$B_n = \sum_{i=1}^{n} S(n,i)$$

Example

Here are the values for $S_2(10,i)$, $i = 0, \ldots, 10$:

$$0, 1, 511, 9330, 34105, 42525, 22827, 5880, 750, 45, 1$$

Hence $B_{10} = 115975$

Lemma

$$
\begin{aligned}
S_2(n,0) &= 0 \qquad \textit{for } n > 0 \\
S_2(n,n) &= 1 \qquad \textit{for } n \geq 0 \\
S_2(n,k) &= k \cdot S_2(n-1,k) + S_2(n-1,k-1)
\end{aligned}
$$

Claim

There are $k!\,S(n,k)$ *surjective functions from* $[n]$ *to* $[k]$.

Proof.

For every surjective function $f : [n] \to [k]$ the kernel equivalence K_f has exactly k classes.

But $K_f = K_g$ iff $f = \pi \circ g$ for some permutation π of $[k]$.

Hence, each partition into k classes corresponds to $k!$ many surjections. □

Falling Factorials

Stirling numbers appear when one tries to rewrite x^n as a polynomial constructed from falling factorials $x^{\underline{k}}$.

Lemma

$$x^n = \sum_{i \leq n} S_2(n, i) \cdot x^{\underline{i}}$$

Naturally one also wants to be able to go in the opposite direction: rewrite $x^{\underline{n}}$ as an ordinary polynomial.

This leads to Stirling numbers of the first kind, written $S_1(n, k)$.

Lemma

$$x^{\underline{n}} = \sum_{i \leq n} (-1)^{n-i} S_1(n, i) \cdot x^i$$

$S_1(n,k)$ is the number of permutations of $[n]$ with exactly k cycles.

Thus, $S_1(n,k)$ is the number of ways an n-set can be partitioned into k cycles (rather than blocks). A cycle is a sequence, but we identify sequences that can be obtained from each other by rotation.

So, as cycles a,b,c,d and c,d,a,b are the same.

Example

$S_1(4,2) = 11$, and the cycle decompositions are

$$\begin{array}{ll} (a),(b,c,d) & (a),(b,d,c) \\ (b),(a,c,d) & (b),(a,d,c) \\ (c),(a,b,d) & (c),(a,d,b) \\ (d),(a,b,c) & (a),(a,c,d) \\ (a,b),(c,d) & (a,c),(b,d) \\ (a,d),(b,c) & \end{array}$$

It follows that

$$\sum_{i \le n} S_1(n,i) = n!$$

Lemma

$$\begin{aligned} S_1(n,0) &= 0 \qquad \textit{for } n > 0 \\ S_1(n,n) &= 1 \qquad \textit{for } n \ge 0 \\ S_1(n,k) &= (n-1) \cdot S_1(n-1,k) + S_1(n-1,k-1) \end{aligned}$$

In this case we are looking for all nondecreasing sequences (n_i) such that

$$\sum_{i=1}^{k} n_i = n$$

where $n_i \geq 0$ in the general case, and $n_i > 0$ when only non-empty boxes are allowed.

This is usually called a partition problem.

Equivalently, we have to find non-negative integer solutions of

$$x_1 + 2x_2 + 3x_3 + \ldots + nx_n = n$$

Here x_i is the number of boxes containing i elements.

Too Hard

Unfortunately, this is rather difficult. Suffice it to say that the number can be obtained as the coefficient of x^n in the power series expansion of

$$\frac{1}{(1-x)(1-x^2)\dots(1-x^n)}$$

Example

$$\prod(1-x)^{-i} = 1 + x + 2\,x^2 + 3\,x^3 + 5\,x^4 + 7\,x^5 + O(x^6)$$

Hence, there are 7 unrestricted partitions of 5.

$$\begin{array}{rl} 1,1,1,1,1 & 1,1,3 \\ 1,1,1,2 & 1,4 \\ 1,2,2 & 5 \\ 2,3 & \end{array}$$

Here is a summary of our results.

objects	boxes	empty	
+	+	+	k^n
+	+	−	$k!\,S_2(n,k)$
+	−	+	B_n
+	−	−	$S_2(n,k)$
−	+	+	$C(n+k-1,k)$
−	+	−	$C(n-1,k-1)$
−	−	+	
−	−	−	

- Counting
- Multinomials

3. Inclusion/Exclusion

How many ways can one rearrange the letters in "wedigmath" so that neither "we" nor "dig" nor "math" appears?

All letters are distinct, so there are $9!$ permutations of the letters. Let U be all these permutations.

Let A_1 all words containing "we", A_2 all words containing "dig", and A_3 all words containing "math".

We want

$$|U| - |A_1 \cup A_2 \cup A_3|$$

But

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| \\ - |B \cap C| + |A \cap B \cap C|$$

$$\begin{aligned} |U| &= 9! \\ |A_1| &= 8! \\ |A_2| &= 7! \\ |A_3| &= 6! \\ |A_1 \cap A_2| &= 6! \\ |A_1 \cap A_3| &= 5! \\ |A_2 \cap A_3| &= 4! \\ |A_1 \cap A_2 \cap A_3| &= 3! \end{aligned}$$

Hence we get

$$9! - 8! - 7! - 6! + 6! + 5! + 4! - 3! = 317658$$

The last example is based on computing the cardinality of a union of sets.

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$\begin{aligned}|A \cup B \cup C| &= |A| + |B| + |C| \\ &\quad - |A \cap B| - |A \cap C| - |B \cap C| \\ &\quad + |A \cap B \cap C|\end{aligned}$$

How does this generalize to $|A_1 \cup A_2 \cup \ldots \cup A_n|$?

We should expect a large, alternating sum involving intersections of k sets, for all $k = 1, \ldots, n$.

Lemma (Sylvester)

Let $A = \{A_1, A_2, \ldots, A_n\}$ *and*
$U = \bigcup A = A_1 \cup A_2 \cup \ldots \cup A_n$.

$$|U| = \sum_{\emptyset \neq B \subseteq A} (-1)^{|B|+1} \left| \bigcap B \right|$$

Note that B here is a family of subsets of U, so $\bigcap B$ is a subset of U.

This theorem can be proved by induction, or by clever manipulations of functions, but we will forego the opportunity.

How many integer solutions are there for

$$x_1 + x_2 + x_3 + x_4 = 40$$
$$0 \leq x_i \leq 15$$

Main line of attack: express as an occupancy problem: place 40 balls into four boxes.

Ignoring the constraint $x_i \leq 15$ there are

$$C(40 + 4 - 1, 4 - 1) = C(43, 3) = 12341$$

solutions $x = (x_1, x_2, x_3, x_4)$.

No good: we must subtract "bad" solutions: that's where Inc/Exc comes in.

Define

$$A_i = \text{ solutions with } x_i \geq 16$$
$$A = \{A_1, A_2, A_3, A_4\}$$
$$U = A_1 \cup A_2 \cup A_3 \cup A_4$$

So U is the set of all bad solutions.

By I/E, we need to compute

$$|U| = \sum_{\emptyset \neq B \subseteq A} (-1)^{|B|+1} \Big| \bigcap B \Big|$$

But we can only have at most 2 bad x_i's in any bad solution x: otherwise we get a sum of at least 48.

Hence $\bigcap B = \emptyset$ for $|B| > 2$.

So, we only need to deal with $B = \{A_i\}$ and $B = \{A_i, A_j\}$.

By symmetry we get $4 \cdot C(27, 3)$ in the first case: there are four choices for i, but the value of i does not matter. Let's assume $i = 1$.

Think of placing 16 balls into x_1, and then distributing the remaining $24 = 40 - 16$ balls into the four boxes. There are $C(24 + 4 - 1, 4 - 1) = C(27, 3)$ ways of doing this.

In the second case we similarly obtain $6 \cdot C(11, 3) = 10710$.

So, the number of solutions is

$$12341 - (11700 - 990) = 1631.$$

Make sure you understand the details, this is a bit tricky.

We already know that the number of surjective functions from $[n]$ to $[k]$ is $k!\, S_2(n,k)$.

Can we avoid Stirling numbers? Sounds very hard, but Inclusion/Exclusion takes care of it. Let

$$A_i = \{\, f : [n] \to [k] \mid i \notin \mathrm{rng}\, f \,\}$$

Note that f is surjective iff $f \notin U = A_1 \cup \ldots \cup A_k$.

Now apply the Inclusion/Exclusion Principle:

$$
\begin{aligned}
k^n - |U| &= k^n - \sum_{\emptyset \neq B \subseteq A} (-1)^{|B|+1} \Big| \bigcap B \Big| \\
&= \sum_{B \subseteq A} (-1)^{|B|} \Big| \bigcap B \Big| \\
&= \sum_{B \subseteq A} (-1)^{|B|} (k - |B|)^n \\
&= \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^n
\end{aligned}
$$

Definition

The nth harmonic number H_n is defined by

$$H_n = 1 + 1/2 + 1/3 + \ldots + 1/n$$

The first few values of H_n are

$$1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \frac{137}{60}, \frac{49}{20}, \frac{363}{140}, \frac{761}{280}, \frac{7129}{2520}, \frac{7381}{2520}$$

From calculus, the series $\sum 1/n$ diverges, so $H_n \to \infty$ as $n \to \infty$.

But divergence is glacially slow: $H_{10000} \approx 9.79$.

The numerator of this fraction has 4346 digits, and the denominator has 4345.

Note that

$$H_n - 1 < \int_1^n \frac{1}{x}\,dx < H_{n-1}$$

so that $0 \leq H_n - \ln n \leq 1$.

One wonders whether $H_n - \ln n$ converges to a particular value as n tends to infinity. Euler showed that the limit indeed exists. Nowadays it is referred to as the Euler-Mascheroni constant and usually written γ. We have

$$\gamma \approx 0.5772156649015328$$

Here is an estimate of convergence:

$$H_n - \ln n = \gamma + \frac{1}{2n} + O(n^{-2})$$

Amazingly, it is not known whether γ is irrational.

Lemma

Let $1 \leq m < n$. *Then* $H_n - H_m$ *is not integral.*

Proof.

Let $1 \leq m < n$. It is not hard to see that the sequence $m, m+1, \ldots, n$ contains a unique element p that maximizes ν_2, say, $p = p_0 2^k$ where p_0 is odd.

Forming the sum of the fractions with denominator lcm$(m, \ldots, n) = 2^k \alpha$, α odd, leads to a numerator of the form $\alpha' + \beta$ where $\alpha' = \alpha / p_0$ is odd, and β is even.

Hence $H_n - H_m$ cannot be an integer.

□

Lemma

The decimal expansion of H_n is non-terminating except for $H_1 = 1$, $H_2 = 1.5$ and $H_6 = 2.45$.

The lemma is easy to verify for, say, $n \leq 100$, but somewhat difficult to prove.

Write $H_n = a_n/b_n$ where the fraction is in lowest common terms. Then one can show that all primes p such that $(n+1)/2 \leq p \leq n$ divide b_n. Use the Bertrand-Chebyshev theorem to show that this guarantees the existence of a prime dividing b_n other than 2 and 5.

E.g. here is the factorization of H_{50}:

$$2^5\,3^3\,5^2\,7^2\,11\,13\,17\,19\,23\,29\,31\,37\,41\,43\,47$$

Here is a little challenge: determine

$$\mathcal{H}_n = \sum_{k \leq n} H_k$$

To get some rough idea what the value of $\mathcal{H}_n$ should be it is a good idea to switch to integrals:

$$\mathcal{H}_n \approx \int_1^n \ln x \, dx = n \ln n - n + 1$$

So an educated first guess would be $\mathcal{H}_n \approx nH_n - n$.

But how do we go about calculating the discrete sum rather than the integral?

$$\begin{aligned}\mathcal{H}_n &= \sum_{k=1}^{n}\sum_{i=1}^{k} 1/i\\ &= \sum_{i=1}^{n} 1/i \sum_{k=1}^{i} 1\\ &= \sum_{i=1}^{n} (n-i+1)/i\\ &= \sum_{i=1}^{n} (n+1)/i \; - n\\ &= (n+1)H_n - n\end{aligned}$$

Not bad at all. So we have $\sum_{k=1}^{n} H_k = (n+1)\cdot H_n - n$.