CDM

Finite Fields

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Where Are We?

We know that every finite field carries two apparently separate structures: additive and multiplicative.

addition	$\mathbb{F} \cong (\mathbb{Z}_p)^k$	$(a_1, \ldots, a_k)$
multiplication	$\mathbb{F}^\times \cong \mathbb{Z}_{p^k-1}$	g^i

The problem is that we have absolutely no idea how to unify the two.

Back to the Roots

Time to get serious about building a finite field.

We would like to follow the construction of $\mathbb{Q}(\sqrt{2})$ from above, adjoining a root of $x^2 - 2 = 0$ to the rationals. But this time, we won't rely on intuition and prior knowledge of the reals. For example, consider the polynomial

$$f = x^2 + x + 1 \in \mathbb{F}_2[x]$$

Note that one can easily check that f has no root over $\mathbb{F}_2$.

So how do we expand $\mathbb{F}_2$ to a field $\mathbb{F}$ where f has a root?

Obstructions

This time:

- We do not know a convenient big field like $\mathbb{R}$ that we can use as a safe sandbox, and
- we have no intuitive idea what a root of f looks like.

So, we can't just do

$$\mathbb{Q}(\sqrt{2}) = \{\, a + b\sqrt{2} \mid a, b \in \mathbb{Q} \,\} \subseteq \mathbb{R}$$

But: we can interpret this construction as the result of applying the simplification rule

$$x^2 \rightsquigarrow 2$$

to polynomials over $\mathbb{Q}$. In this setting, the "unknown" x works just like the root we are after.

Generalizing $\sqrt{2}$

So the hope is that we can generalize this idea by starting with $\mathbb{F}_2[x]$ and we use the simplification rule

$$x^2 \rightsquigarrow x + 1$$

Recall, we are dealing with characteristic 2, so plus is minus.

By systematically applying this rule, plus standard field arithmetic, we might be able to construct a finite field that has a root for f.

The Rewrite Rule

So what happens to $\mathbb{F}_2[x]$ if we apply this rule systematically? Here is a (characteristic) example.

$$\begin{aligned} x^6 + x^3 + x + 1 &\rightsquigarrow (x+1)^3 + x(x+1) + x + 1 \\ &\rightsquigarrow (x^3 + x^2 + x + 1) + (x^2 + x) + x + 1 \\ &\rightsquigarrow x(x+1) + (x+1) + 1 \\ &\rightsquigarrow x + 1 \end{aligned}$$

In other words, $x^6 + x^3$ reduces to 0 (check).

It is easy to see that any polynomial will similarly ultimately produce a polynomial of degree at most 1: any higher order term could be further reduced.

Simplification, Algorithmically

In general, if we start with a polynomial of degree d we can reduce everything down to polynomials of degree at most $d-1$. Since the coefficients come from a finite field there are only finitely many such polynomials; in fact there are q^d where q is the size of the field.

But we don't just get a bunch a of polynomials, we also get the operations that turn them into a field:

- Addition is simply addition of polynomials in $\mathbb{F}[x]$.
- Multiplication is multiplication of polynomials in $\mathbb{F}[x]$ followed by a reduction: we have to apply the simplification rule until we get back to a polynomial of degree less than d.

Not bad, but we need a more elegant algebraic description of this process.

Simplification, Algebraically

Obviously, the simplification rule $x^2 \rightsquigarrow x + 1$ has lots of algebraic consequences.

In order to really get a grip on these, we need to determine all equational consequences of the identity $x^2 = x + 1$. It is not hard to see that we obtain an equivalence relation on polynomials in $\mathbb{F}_2[x]$ that is nicely compatible with the structure of the ring (a congruence).

Given a congruence, we can form a quotient ring, which turns out to be exactly the field we are looking for.

The key idea here is to consider so-called ideals in a ring (which are related to but different from subrings).

Ideals

Definition

Let R be a commutative ring. An ideal $I \subseteq R$ is a subset that is closed under addition and under multiplication by arbitrary ring elements: $a \in I, b \in R$ implies $ab \in I$.

So an ideal is much more constrained than a subring: it has to be closed by multiplication from the outside. Ideals are hugely important since they produce congruences and thus allow us to form a quotient structure:

$$a = b \pmod{I} \qquad \text{iff} \qquad a - b \in I.$$

As a consequence, arithmetic in this quotient structure is well-behaved: E.g.

$$a = a', b = b' \pmod{I} \quad \Rightarrow \quad a + b = a' + b', a\,b = a'b' \pmod{I}$$

Generating Ideals

What is the smallest ideal containing elements $a_1, \ldots, a_k \in R$?

All we need is linear combinations: the ideal generated by $a_1, \ldots, a_k$ is

$$(a_1, \ldots, a_k) = \{\, r_1 a_1 + \ldots + r_k a_k \mid r_i \in R \,\}$$

In particular for $k = 1$ we have

$$(a) = \{\, ra \mid x \in R \,\}$$

This is the principal ideal generated by a.

The ideals $\{0\}$ and R are called trivial, all others are proper.

Note that a field is a commutative ring that has no proper ideals.

Modular Arithmetic Revisited

Familiar example: to describe modular arithmetic we have $R = \mathbb{Z}$ as ground ring and use the principal ideal $I = (m) = m\,\mathbb{Z}$.

$$a = b \pmod{I} \qquad \text{iff} \qquad m \,\big|\, (a - b).$$

The ideal I is generated by the modulus $m > 0$ and the quotient ring $\mathbb{Z}_m = \mathbb{Z}/(m)$ is finite in this case.

Moreover, the quotient ring $\mathbb{Z}_m$ is a field iff m is prime.

Principal Ideal Domains

Definition

A principal ideal domain (PID) is an integral domain, all of whose ideals are principal.

Important examples of PIDs are

- the integers $\mathbb{Z}$ (think GCD)
- the Gaussian integers $\mathbb{Z}[\mathbf{i}]$
- a polynomial ring $\mathbb{F}[x]$ where $\mathbb{F}$ is a field

Counterexamples: $\mathbb{Z}[x]$ and $\mathbb{F}[x, y]$ both fail to be PIDs.

Minimal Polynomials

Suppose we have an extension $\mathbb{F} \subseteq \mathbb{K}$ with $\alpha \in \mathbb{K}$ algebraic over $\mathbb{F}$. Let

$$I = \{ f \in \mathbb{F}[x] \mid f(\alpha) = 0 \}$$

Then I is an ideal and we must have $I = (g)$.

The polynomial g has minimal degree among all the annihilators of α, and we may safely assume that g is monic.

Definition

This polynomial g is the minimal polynomial of α over $\mathbb{F}$.

Digression: Proper Substructures

In algebra it is important to come up with the right notion of substructure: just picking a subset that is closed under the algebraic operations is often not enough.

- For groups, normal subgroups are arguably more important than plain subgroups.
- For rings, ideals are arguably more important than subrings.
- But a sub-vectorspace is just the right notion.

Irreducible Polynomials

Ideals provide the right type of equivalence relation for the construction of a finite field from a polynomial ring. Alas, the ideals cannot be chosen arbitrarily, we need to start from special polynomials, in analogy to m being prime in the integer case.

Definition

A polynomial is irreducible if it is not the product of polynomials of smaller degree.

Irreducibility is necessary when we try to construct a field $\mathbb{F}[x]/(f)$: otherwise we do not even get an integral domain.

For suppose $f(x) = f_1(x)f_2(x)$ where both f_1 and f_2 have degree at least 1. Then $1 \leq \deg(f_i) < \deg(f)$, so neither f_1 or f_2 can be simplified in $\mathbb{F}[x]/(f)$.

In particular both elements in $\mathbb{F}[x]/(f)$ are non-zero, but their product is zero.

Counting Irreducibles

Question: How many irreducible polynomials of degree k are there in $\mathbb{F}_2[x]$?

Let's write I_k for this number, so trivially $0 \leq I_k < 2^k$.

Proposition

$$\frac{1}{1-2z} = \prod_{m \geq 1} \left(\frac{1}{1-z^m} \right)^{I_m}$$

This comes down to the simple observation that every polynomial is a product of powers of irreducible ones, and this decomposition is essentially unique: if we order the factors in some canonical way, then the decomposition is unique.

Exercise

Prove the proposition using the hint above.

Möbius Inversion

A bit more fumbling shows that

Lemma

Let I_m be the number of irreducible polynomials in $\mathbb{F}_2[x]$ of degree m. Then

$$2^m = \sum_{d \mid m} d\, I_d.$$

We can apply Möbius inversion to express I_m as a sum. Recall the Möbius function μ:

$$\mu(n) = \begin{cases} +1 & \text{if } n \text{ square-free, even number of prime factors} \\ -1 & \text{if } n \text{ square-free, odd number of prime factors} \\ 0 & \text{otherwise.} \end{cases}$$

Flipping the Sum

$$I_m = \frac{1}{m} \sum_{d|m} \mu(m/d)\, 2^d.$$

The last expression is fairly elegant, but it's not clear what one should expect numerically. Here are the first 20 values of I_m.

1−5	2	1	2	3	6
6−10	9	18	30	56	99
11−15	186	335	630	1161	2182
16−20	4080	7710	14532	27594	52377

Incidentally, $I_{50} = 22517997465744$, about 2 percent.

General Characteristic

The result easily carries over the fields of characteristic p.

Lemma

Suppose $\mathbb{F}$ is a finite field of cardinality q. Then the number of irreducible polynomials in $\mathbb{F}[x]$ of degree d is

$$N_m^q = \frac{1}{m} \sum_{d|m} \mu(m/d) \cdot q^d$$

Modular Arithmetic for Polynomials

Suppose $\mathbb{F}$ is a field and consider an irreducible polynomial $f(x)$ and the principal ideal $(f(x)) = f(x)\mathbb{F}[x]$ that it generates.

We identify two polynomials when their difference is divisible by f:

$$h(x) = g(x) \pmod{f(x)} \quad \Longleftrightarrow \quad f(x) \mid (h(x) - g(x))$$

Let d be the degree of f. Then any polynomial h is equivalent to a polynomial g of degree less than d: write $h(x) = q(x)f(x) + g(x)$ by polynomial division.

In other words, $g(x) = h(x) \bmod f(x)$ is the normal form we are looking for.

A Harder Example

Over $\mathbb{F}_2$, the polynomial

$$f(x) = x^3 + x + 1$$

is irreducible.

Again, unlike with the earlier square root over $\mathbb{R}$ example, it is absolutely not clear what a root of $f(x) = 0$ should look like.

In order to manufacture a root, we want to compute in the polynomial ring $\mathbb{F}_2[x]$ modulo the ideal $I = (f(x))$ generated by f.

There are two steps:

- Find a good representation for $\mathbb{F}_2[x]/I$.
  Comes down to picking a representative in each equivalence class.
- Determine how to perform addition, multiplication and division on these representatives.

Representatives

Note that

$$x^3 = x + 1 \pmod{I}$$

(we have characteristic 2, so minus is plus). We can think of this as a rewrite rule that eliminates all monomials of degree at least 3:

$$x^{3k+r} \rightsquigarrow (x+1)^k x^r.$$

By applying this substitution repeatedly, we wind up with 8 polynomials modulo I, namely all the polynomials of degree at most 2:

$$\mathbb{K} = \{0, 1, x, 1+x, x^2, 1+x^2, x+x^2, 1+x+x^2\}$$

This method is very simple, but slow. In reality one would use polynomial division to get the remainders mod f.

The Magic Root

We write α for (the equivalence class of) x for emphasis, $\alpha = x \bmod f(x)$.

Then $\alpha \in \mathbb{K}$ really is a root of $f(x) = 0$ in the extension field $\mathbb{K}$.

Why? We have by brute force

$$f(\alpha) = x^3 + x + 1 = 0 \pmod{I}$$

Yes, this is a bit lame. One would have hoped for some kind of fireworks, some clever way of writing down the root in terms of some fancy polynomial.

But, it's really no different from the $\sqrt{2}$ example, just less familiar.

Representatives II

Again, algebraically, it is best to think of the extension field $\mathbb{F}_2 \subseteq \mathbb{K}$ as a quotient structure, as the polynomials modulo f:

$$\mathbb{K} = \mathbb{F}_2[x]/(f(x))$$

With a view towards algorithms, we can make things more combinatorial by keeping track of coefficient vectors, in this case

$$c_2x^2 + c_1x + c_0 \rightsquigarrow (c_2, c_1, c_0)$$

where $c_i \in \mathbb{F}_2$ is just a single bit.

Addition, Algorithmically

In this setting the additive structure trivial: it's just componentwise addition of these triples mod 2.

$$(c_2, c_1, c_0) + (c_2', c_1', c_0') = (c_2 + c_2', c_1 + c_1', c_0 + c_0')$$

As observed before, the additive group of these fields is just a Boolean group. Note that this operation is trivial to implement (xor on bit-vectors, can even be done in 32 or 64 bit blocks).

For other characteristics, though, we have to use modular numbers.

Multiplication, Algorithmically

How about multiplication? Since multiplication increases the degree, we can't just multiply out, but we have to simplify using our rule $x^3 \to x + 1$ afterwards.

The product

$$(c_2, c_1, c_0) \cdot (c_2', c_1', c_0') = (d_2, d_1, d_0)$$

is given by the coefficient triple

$$\begin{aligned} d_2 &= c_2\,c_0' + c_1\,c_1' + c_0\,c_2' + c_2\,c_2' \\ d_1 &= c_1\,c_0' + c_0\,c_1' + c_2\,c_1' + c_1\,c_2' + c_2\,c_2' \\ d_0 &= c_0\,c_0' + c_2\,c_1' + c_1\,c_2' \end{aligned}$$

This is a bit messy, and it gets more messy when we deal with larger degree polynomials. Still, we could hard-wire a circuit.

Feedback-Shift-Registers

It is slightly more elegant to break up the polynomial multiplication into steps involving just multiplication by x. Let f have degree d, say $f(x) = x^d + \sum_{i<d} x^i$. When multiplying $g(x) = \sum_{i<d} c_i x^i$ we get

- for $i < d-1$: $x^i \rightsquigarrow x^{i+1}$
- for $i = d-1$: $x^{d-1} \rightsquigarrow x^d \rightsquigarrow \sum_{i<d} x^i$

But that's exactly what a FSR (albeit of Galois type) does: shift all the bits, and flip bits according to some fixed pattern.

So FSR just compute multiplication in a finite field of characteristic 2. That is the central reason why we can analyze them.

Multiplicative Structure

Recall that α is the equivalence class of x. In our example, the powers of α are:

$$\begin{aligned} \alpha^0 &= 1 && = (0,0,1) \\ \alpha^1 &= \alpha && = (0,1,0) \\ \alpha^2 &= \alpha^2 && = (1,0,0) \\ \alpha^3 &= \alpha + 1 && = (0,1,1) \\ \alpha^4 &= \alpha^2 + \alpha && = (1,1,0) \\ \alpha^5 &= \alpha^2 + \alpha + 1 && = (1,1,1) \\ \alpha^6 &= \alpha^2 + 1 && = (1,0,1) \end{aligned}$$

So α is the generator of $\mathbb{F}^\times$.

Note: this does not always work, the order of α may be strictly less than $2^d - 1$.

Table of Inverses

We really obtain a field this way, not just a ring (recall that f is irreducible).

	h	h^{-1}
1	1	1
2	α	$1 + \alpha^2$
3	α^2	$1 + \alpha + \alpha^2$
4	$1 + \alpha$	$\alpha + \alpha^2$
5	$1 + \alpha^2$	α
6	$\alpha + \alpha^2$	$1 + \alpha$
7	$1 + \alpha + \alpha^2$	α^2

Note that this table defines an involution: $(h^{-1})^{-1} = I$.

The Structure of Finite Fields

Recall the big theorem we announced some time ago:

Theorem

Every finite field $\mathbb{F}$ has cardinality p^k where p is prime and the characteristic of $\mathbb{F}$, and $k \geq 1$. Moreover, for every p prime and $k \geq 1$ there is a finite field of cardinality p^k and all fields of cardinality p^k are isomorphic.

So there are three assertions to prove:

- Every finite field $\mathbb{F}$ has cardinality p^k where p is the characteristic of $\mathbb{F}$ and therefore prime.
- There is a field of cardinality p^k.
- All fields of cardinality p^k are isomorphic.

Proof Sketch

We have already taken care of parts 1 and 2:

- Since $\mathbb{F}$ is finite vector space over $\mathbb{Z}_p$ where p is the characteristic of $\mathbb{F}$ it must have size p^k, p prime, $k \geq 1$.
- Since there are irreducible polynomials over $\mathbb{Z}_p$ of degree k for any k we can always construct a finite field of the form $\mathbb{Z}_p[x]/(f)$ of size p^k.

The Problem:
It is absolutely unclear that all these quotient rings are isomorphic.

Battleplan

Issue 1 Suppose we use some irreducible polynomial f. Say f has roots α and β. Why should we have

$$\mathbb{F}_p(\alpha) \cong \mathbb{F}_p(\beta)$$

Issue 2 Suppse f and g are two irreducible polynomials of the same degree. Why should we have

$$\mathbb{F}_p[x]/(f) \cong \mathbb{F}_p[x]/(g)$$

We will first deal with issue 1, and then extend the method to handle issue 2.

First a few handy tools.

Homomorphisms and Kernels

Let's collect some tools to compare rings and fields.

Definition

Let R and S be two rings and $f : R \to S$. f is a ring homomorphism if

$$f(g+h) = f(g) + f(h) \quad \text{and} \quad f(gh) = f(g)f(h).$$

If f is in addition injective/surjective/bijective we speak about monomorphisms, epimorphism and isomorphisms, respectively. The kernel of a ring homomorphism is the set of elements that map to 0.

Notation: $\ker(f)$.

Note that $f(0) = 0$. Since $f(x) = f(y)$ iff $x - y \in \ker(f)$ a ring homomorphism is a monomorphism iff its kernel is trivial: $\ker(f) = \{0\}$.

It is easy to see that the kernel of any ring homomorphism $f : R \to S$ is an ideal in R.

Rings with 1

Recall that our rings always have a multiplicative unit (as opposed to abominable rngs). So one requires

$$f(1) = 1.$$

These are sometimes called unital ring homomorphisms. In particular field homomorphisms are unital.

Lemma

If $f : \mathbb{F} \to \mathbb{K}$ is a field homomorphism, then f is injective.

Proof.

$\ker(f) \subseteq \mathbb{F}$ is an ideal. But in a field there are only two ideals: $\{0\}$ and the whole field. Since $f(1) = 1$, 1 is not in the kernel, so the kernel must be $\{0\}$ and f is injective.

□

The Frobenius Homomorphism

Here is a somewhat surprising example of a homomorphism.

Definition

Let R be a ring of characteristic $p > 0$. The Frobenius homomorphism is defined by the map $R \to R$, $x \mapsto x^p$.

The Frobenius map is indeed a ring homomorphism since R has characteristic p:

$$(a+b)^p = a^p + b^p.$$

Over a finite field we even get an automorphism. The orbits of a non-zero element look like

$$a, a^p, a^{p^2}, \ldots, a^{p^{k-1}}$$

Exercise

Use the binomial theorem to prove that the Frobenius map is a homomorphism.

Aside: Galois

We are now very close to Galois theory, the study of automorphism groups of fields initiated by Evariste Galois (1811-1832).

$1832 - 1811 = 21$: a brilliant mathematician, but a bad shot.

More Aside

For algebra lovers: the Frobenius homomorphism F is the key to understanding the Galois group of the algebraic closure K of a finite field $\mathbb{F}_p$.

Recall that the Galois group is the collection of all automorphisms of K that leave the prime field invariant. By Fermat's little theorem, $\mathbb{F}_p$ is certainly invariant under F.

Algebraically closed fields are perfect, so F is indeed an automorphism of K. The group generated by F is a subgroup of the Galois group, the so-called Weil group. In fact, the whole Galois group is a kind of completion of the Weil group.

Adjoining Different Roots

Back to our uniqueness problem. As a first step, consider two roots of the same polynomial. More precisely, let $f(x) \in \mathbb{F}[x]$ irreducible, $\alpha_1 \neq \alpha_2$ two roots of f. Consider two corresponding simple field extensions $\mathbb{F} \subseteq \mathbb{K}_i = \mathbb{F}(\alpha_i)$ where $\alpha_i \in \mathbb{K}_i$.

Theorem (Extension Isomorphism Theorem)

There is a unique isomorphism $\varphi : \mathbb{K}_1 \to \mathbb{K}_2$ such that $\varphi(\alpha_1) = \alpha_2$ and φ is the identity on $\mathbb{F}$.

Proof.

We exploit the fact that $\mathbb{K}_i = \mathbb{F}[\alpha_i]$ and define

$$\varphi(p(\alpha_1)) = p(\alpha_2)$$

for any $p \in \mathbb{F}[x]$.

Proof

It is straightforward but tedious to check that φ has all the right properties.

The interesting part is to verify well-definedness.

To this end, suppose $p(\alpha_1) = p'(\alpha_1)$.
Then f divides $p - p'$, say $p - p' = f \cdot q$.
But then $(p - p')(\alpha_2) = f(\alpha_2) \cdot q(\alpha_2) = 0$.

□

Exercise

Work out the rest of the proof.

Establishing Isomorphisms

The following fact is often useful to establish an isomorphism. Suppose $f : R \to S$ is an epimorphism (no major constraint, otherwise replace S by the range of f). Then $R/\ker(f)$ is isomorphic to S.

For example, we can use this technique to prove our old theorem about field extensions by adjoining roots.

More precisely, let $\mathbb{F}(\alpha)$ be the smallest field $\mathbb{F} \subseteq \mathbb{F}(\alpha) \subseteq \mathbb{K}$ that contains a root $\alpha \in \mathbb{K}$ of some polynomial $f \in \mathbb{F}[x]$. Then

$$\mathbb{F}(\alpha) = \{\, g(\alpha) \mid g \in \mathbb{F}[x] \,\} = \mathbb{F}[\alpha]$$

rather than, say, the collection of rational functions over $\mathbb{F}$ evaluated at α.

Adjoining a Root, contd.

To see why, note that the right hand side is the range of the evaluation map

$$\begin{aligned} \nu\colon \mathbb{F}[x] &\longrightarrow \mathbb{K} \\ g &\mapsto g(\alpha) \end{aligned}$$

that evaluates g at α, producing a value in $\mathbb{K}$. It is easy to check that ν is a ring homomorphism and clearly $(f) \subseteq \ker(\nu)$.

We may safely assume that f is monic and has minimal degree in $\mathbb{F}[x]$ of all polynomials with root α. Then f is irreducible and we have

$$\ker(\nu) = \{\, p \in \mathbb{F}[x] \mid f \text{ divides } p \,\} = (f)$$

This shows that the range of ν is isomorphic to $\mathbb{F}[x]/(f)$ and hence a field.

Irreducibility is essential here, otherwise $f(x) = (x^2-2)(x^2-3) = x^4 - 5x^2 + 6$ with $\alpha = \sqrt{2}$ over $\mathbb{F} = \mathbb{Q} \subseteq \mathbb{C} = \mathbb{K}$ would produce a non-integral domain.

Kernels, The Idea

Note that this is the third time we encounter kernels.

- For a general function $f : A \to B$ the kernel relation is given by $f(x) = f(y)$.
- For a group homomorphism $f : A \to B$ the kernel is given by $\{\, x \in A \mid f(x) = 1 \,\}$.
- For a ring homomorphism $f : A \to B$ the kernel is given by $\{\, x \in A \mid f(x) = 0 \,\}$.

In the last two cases we can easily recover the classical kernel relation and the definition as stated turns out to be more useful.

Still, there is really just one idea.

Uniqueness

Back to the problem of showing that there is only "one" finite field $\mathbb{F}_{p^k}$ of size p^k. To understand finite fields completely we need just one more idea.

Definition

Let $f \in \mathbb{F}[x]$ monic, $\mathbb{F} \subseteq \mathbb{K}$. Field $\mathbb{K}$ is a splitting field of f if

- $f(x) = (x - \alpha_1) \dots (x - \alpha_d)$ in $\mathbb{K}[x]$, and
- $\mathbb{K} = \mathbb{F}(\alpha_1, \dots, \alpha_d)$.

Needless to say, the $\alpha_i \in \mathbb{K}$ are exactly the roots of f. Thus, in a splitting field we can decompose the polynomial into linear factors.

In other words, $\mathbb{K}$ is the smallest field where f splits into linear factors; by adjoining all the roots of f we get all of $\mathbb{K}$.

Examples

Example

$\mathbb{C}$ is the splitting field of $x^2 + 1 \in \mathbb{R}[x]$.

It is hugely surprising that over $\mathbb{C}$ any non-constant real polynomial can already be decomposed into linear factors, everybody splits already.

Example

Consider $f(x) = x^8 + x \in \mathbb{F}_2[x]$. Then

$$f(x) = x(x+1)(x^3 + x^2 + 1)(x^3 + x + 1)$$

Adjoining one root of $g(x) = x^3 + x + 1$ already produces the splitting field of f: the other irreducible factor of degree 3 also splits.

Example contd.

$$x^8 + x = x(x+1)(x^3 + x^2 + 1)(x^3 + x + 1)$$

element	root of
0	x
α^0	$x + 1$
α^1	$x^3 + x + 1$
α^2	$x^3 + x + 1$
α^3	$x^3 + x^2 + 1$
α^4	$x^3 + x + 1$
α^5	$x^3 + x^2 + 1$
α^6	$x^3 + x^2 + 1$

Splitting Field theorem

Our next goal is to establish the following result.

Theorem (Splitting Field Theorem)

For any irreducible polynomial there exists a splitting field, and any two such splitting fields are isomorphic.

Note that we have all the tools to construct a splitting field: we just keep adjoining roots of irreducible factors of the given polynomial.

But for the uniqueness part we need a bit more machinery.

Basic problem: what would happen in the last example if we had chosen $x^3 + x + 1$ rather than $x^3 + x^2 + 1$? We get isomorphic vector spaces, but why should the multiplicative structure be the same?

Constructing Isomorphisms

Suppose we have an isomorphism $\theta : \mathbb{F}_1 \to \mathbb{F}_2$ and $f_1 \in \mathbb{F}_1[x]$. Set $f_2 = \theta(f_1)$ and let $\mathbb{F}_i \subseteq \mathbb{K}_i$ be splitting fields for f_i.

Lemma

There exists a isomorphism $\varphi : \mathbb{K}_1 \to \mathbb{K}_2$ such that $\varphi \restriction \mathbb{F}_1 = \theta$.

Proof.

We construct φ by repeated application of the Extension Isomorphism theorem.

Technically, we perform induction on the degree $d = [\mathbb{K}_1 : \mathbb{F}_1]$ of the splitting extension.

The case $d = 1$ is trivial, since then $\mathbb{F}_i = \mathbb{K}_i$.

So suppose $d > 1$. We may safely assume that f is not irreducible; otherwise the Extension Isomorphism theorem suffices.

Since $\mathbb{F}_1 \neq \mathbb{K}_1$ there is some irreducible factor $g_1 \in \mathbb{F}[x]$ of f_1 of degree larger than 1.

g_1 splits in $\mathbb{K}_1$, say, it has a root α_1.

Set $g_2 = \theta(g_1)$ with root α_2 in $\mathbb{K}_2$.

By the Extension Isomorphism theorem from above, there is a unique isomorphism $\theta' : \mathbb{F}_1[\alpha_1] \to \mathbb{F}_2[\alpha_2]$ such that $\theta'(\alpha_1) = \alpha_2$ and $\theta' \restriction \mathbb{F}_1 = \theta$.

But $[\mathbb{K}_i[\alpha_i] : \mathbb{F}_i] = \deg g_i < \deg f_i$, so by the induction hypothesis we are done. □

Corollary

Splitting fields are unique up to isomorphism.

Finite Fields Explained

Now we can pin down the structure of all finite fields: they are splitting fields (and hence uniquely determined).

Theorem

There is a unique (up to isomorphism) finite field of size p^k.

Proof.

Let $q = p^k$ and consider $f = x^q - x \in \mathbb{F}_p[x]$.

f has q roots, which form a field. For let a and b two roots, then:

$$f(a+b) = (a+b)^q - (a+b) = a^q - a + b^q - b = 0$$

$$f(ab) = (ab)^q - (ab) = a^q b^q - ab = 0$$

Hence the roots of f form the whole splitting field of f. By the Splitting Field theorem, this field is unique up to isomorphism. □

Fixed Point Perspective

Note that the prime subfield of $\mathbb{F}_q$, $q = p^k$, consists of all fixed points of the Frobenius morphism: $x = x^p$, or $x^p - x = 0$.

Similarly, the whole field can be construed as the fixed points of the map $x \mapsto x^q$, an iterated version of the Frobenius morphism:

$$x^q - x = 0$$

Example: A Splitting Field

We can exploit knowledge of the splitting field to study roots of irreducible polynomials.

Consider the irreducible $f = x^3 + x^2 + 1 \in \mathbb{F}_2[x]$.

Let α be a root of f in the splitting field $\mathbb{K}$. By long division (exploiting $\alpha^3 + \alpha^2 + 1 = 0$) we find all the roots of f:

$$f = (x + \alpha)(x + \alpha^2)(x + \alpha^2 + \alpha + 1)$$

Hence

$$\mathbb{K} = \mathbb{F}_2(\alpha) = \{\, a\alpha^2 + b\alpha + c \mid a, b, c \in \mathbb{F}_2 \,\}$$

Example: The Field $\mathbb{F}_{5^2}$

Consider characteristic $p = 5$ and $k = 2$.

$$\begin{aligned} x^{25} - x = x\,&(1+x)\,(2+x)\,(3+x)\,(4+x) \\ &(2+x^2)\,(3+x^2)\,(1+x+x^2)\,(2+x+x^2)\,(3+2\,x+x^2)\,(4+2\,x+x^2) \\ &(3+3\,x+x^2)\,(4+3\,x+x^2)\,(1+4\,x+x^2)\,(2+4\,x+x^2) \end{aligned}$$

The factorization of $x^{25} - x$ was done by an algorithm, one of the great success stories of computer algebra dating back to the 1960s.

At any rate, there are 10 irreducible quadratic polynomials to choose from.

Which one should we pick?

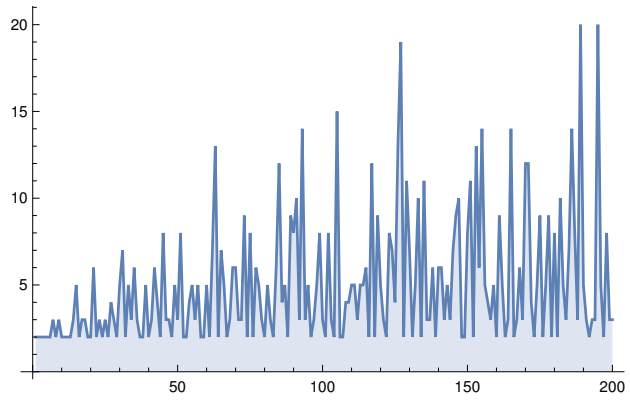
Example: The Field $\mathbb{F}_{11^2}$

The factorization of $X^{11^2} - x$.

$$x(1+x)(2+x)(3+x)(4+x)(5+x)(6+x)(7+x)(8+x)(9+x)(10+x)$$
$$(1+x^2)\,(3+x^2)\,(4+x^2)\,(5+x^2)\,(9+x^2)\,(1+x+x^2)\,(4+x+x^2)\,(6+x+x^2)$$
$$(7+x+x^2)\,(8+x+x^2)\,(2+2x+x^2)\,(4+2x+x^2)\,(5+2x+x^2)\,(6+2x+x^2)$$
$$(10+2x+x^2)\,(3+3x+x^2)\,(6+3x+x^2)\,(8+3x+x^2)\,(9+3x+x^2)\,(10+3x+x^2)\,(2+4x+x^2)$$
$$(5+4x+x^2)\,(7+4x+x^2)\,(8+4x+x^2)\,(9+4x+x^2)\,(1+5x+x^2)\,(2+5x+x^2)\,(3+5x+x^2)$$
$$(7+5x+x^2)\,(10+5x+x^2)\,(1+6x+x^2)\,(2+6x+x^2)\,(3+6x+x^2)\,(7+6x+x^2)$$
$$(10+6x+x^2)\,(2+7x+x^2)\,(5+7x+x^2)\,(7+7x+x^2)\,(8+7x+x^2)\,(9+7x+x^2)\,(3+8x+x^2)$$
$$(6+8x+x^2)\,(8+8x+x^2)\,(9+8x+x^2)\,(10+8x+x^2)\,(2+9x+x^2)\,(4+9x+x^2)$$
$$(5+9x+x^2)\,(6+9x+x^2)\,(10+9x+x^2)\,(1+10x+x^2)\,(4+10x+x^2)\,(6+10x+x^2)$$
$$(7+10x+x^2)\,(8+10x+x^2)$$

This time, there are 55 quadratic irreducible polynomials to pick from.

Number of Terms



Number of terms in the factorization of $x^n - x$ modulo 2, for $n \leq 200$.

Total Recall: Euler's Totient Function

$$\Phi(n) = |\mathbb{Z}_n{}^\star| = |\{\, x < n \mid \gcd(x,n) = 1 \,\}|$$

Euler proved the following product formula

$$\Phi(n) = n \prod_{p|n} (1 - 1/p)$$

where the product is over all primes dividing n.

This is easy to compute given the prime factorization of n, not so easy otherwise.

Primitive Polynomials

Definition

Let $\mathbb{F}$ be a finite field and $f \in \mathbb{F}[x]$ irreducible. f is primitive if $x \bmod f$ is a generator of the multiplicative subgroup in the extension field $\mathbb{F}[x]/(f)$. The roots of a primitive polynomial are also called primitive.

The size of the multiplicative subgroup $\mathbb{F}_q{}^\times$ is $q-1$, $q = p^k$, and we know that the group is cyclic.

Hence there must be $\Phi(q-1)$ generators in this subgroup, corresponding to the number of primitive polynomials.

Primitive Polynomials

Since any of the roots of a corresponding primitive polynomial is a generator, the number of primitive polynomials of degree k is

$$\frac{\Phi(q-1)}{k}$$

For example, in the case $p = 5$, $k = 2$ there are 8 primitive elements and 4 polynomials.

The Order of a Polynomial

There is an alternative way to describe primitive polynomials that avoids references to the extension field construction.

Definition

Let $f \in \mathbb{F}[x]$ such that $f(0) \neq 0$. The order or exponent of f is the least $e \geq 1$ such that f divides $x^e - 1$.

In other words, $x^e = 1 \bmod f$.

So an irreducible f is primitive iff it has order $p^k - 1$ where p is the characteristic and k the degree of f.

The Field $\mathbb{F}_{5^2}$, contd.

For example, $f = 2 + 4\,x + x^2$ is primitive.

α	x	α^{13}	$4\,x$
α^2	$3 + x$	α^{14}	$2 + 4\,x$
α^3	$3 + 4\,x$	α^{15}	$2 + x$
α^4	$2 + 2\,x$	α^{16}	$3 + 3\,x$
α^5	$1 + 4\,x$	α^{17}	$4 + x$
α^6	2	α^{18}	3
α^7	$2\,x$	α^{19}	$3\,x$
α^8	$1 + 2\,x$	α^{20}	$4 + 3\,x$
α^9	$1 + 3\,x$	α^{21}	$4 + 2\,x$
α^{10}	$4 + 4\,x$	α^{22}	$1 + x$
α^{11}	$2 + 3\,x$	α^{23}	$3 + 2\,x$
α^{12}	4	α^{24}	1

So $\mathbb{F}_{5^2}^*$ is indeed cyclic with generator α, and $\mathbb{F}_{5^2}$ has dimension 2 as a vector space over $\mathbb{F}_5$, as required.

Back To Feedback-Shift-Registers

The reason this works is that for a Galois feedback-shift-registers one step very closely corresponds to multiplication by α: shifting means multiply each term by α, then reduce according to the corresponding primitive polynomial f.

If α is a primitive element, $\mathbb{F}_{p^k} = \mathbb{F}_p(\alpha)$, then the FSR essentially just computes the sequence

$$1, \alpha, \alpha^2, \alpha^3, \ldots, \alpha^{p^k-2}, \alpha^{p^k-1} = 1, \alpha, \ldots$$

Unleash the hounds of algebra ...

Still Unhappy?

A lot is known about primitive polynomials: $904,000$ hits on google.

There are tables Hansen.

There are well-analyzed algorithms:

> Nirmal R. Saxena & Edward J. McCluskey
> Primitive Polynomial Generation Algorithms–Implementation and Performance Analysis
> CRC 2004

Trinomials

A lot is known about special-form polynomials:

> Richard P. Brent, Paul Zimmermann
> Twelve new primitive binary trinomials
> arXiv 2016

New primitive polynomials over $\mathbb{F}_2$ of degree $42,643,801$; $43,112,609$; and $74,207,281$.

The Finite Field Bibles

> E. R. Berlekamp
> Algebraic Coding Theory
> McGraw-Hill, 1968.

> R. Lidl, H. Niederreiter
> Introduction to Finite Fields and their Applications
> Cambridge University Press, 1986.