Computational Semantics of Cartesian Cubical Type Theory

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Dependent type theory

Dependent type theory is a language for mathematical reasoning.

Implemented in many proof assistants (software for developing and checking proofs) — Agda, Coq, Lean, Nuprl, . . .

- Four color theorem [Gonthier 2008]
- Feit–Thompson theorem [Gonthier *et al.* 2013]

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- Four color theorem [Gonthier 2008]
- Feit–Thompson theorem [Gonthier *et al.* 2013]
- CompCert C compiler [Leroy 2009]
- Multi-Paxos [Schiper *et al.* 2014]
- mbedTLS HMAC-DRBG [Ye *et al.* 2017]

Dependent type theory

Proof assistants reduce mathematical proofs to primitive inferences.

$$\frac{\Gamma, a : A \vdash M : B(a)}{\Gamma \vdash \lambda a.M : (a : A) \longrightarrow B(a)} \qquad \frac{\Gamma \vdash M : (a : A) \longrightarrow B(a) \qquad \Gamma \vdash N : A}{\Gamma \vdash M\, N : B(N)}$$

Dependent type theory

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$$\frac{\Gamma, a : A \vdash M : B(a)}{\Gamma \vdash \underbrace{\lambda a.M : (a : A) \longrightarrow B(a)}_{\text{dependent function}}} \qquad \frac{\Gamma \vdash M : (a : A) \longrightarrow B(a) \qquad \Gamma \vdash N : A}{\Gamma \vdash M\,N : B(N)}$$

$(n : \mathbb{N}) \longrightarrow \mathsf{List}(n)$

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dependent function

$(n : \mathbb{N}) \to \mathsf{List}(n)$

proof of $\forall a \in A.B(a)$

$\forall n \in \mathbb{N}.\mathsf{isEven}(2n)$

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dependent function $\qquad (n : \mathbb{N}) \to \mathsf{List}(n)$

proof of $\forall a \in A. B(a) \qquad \forall n \in \mathbb{N}.\mathsf{isEven}(2n)$

Dependent types play the role of both sets (*e.g.*, functions) and propositions (*e.g.*, $\forall$).

Identity types

$\mathrm{Id}_A(a, b)$ — a and b are equal in A [Martin-Löf 1975].

- Equality is reflexive — $\mathrm{refl}(a) : \mathrm{Id}_A(a, a)$
- "Everything respects equality" — if it holds for $\mathrm{refl}(a)$, it holds for any $p : \mathrm{Id}_A(a, b)$

Identity types

Everything respects equality $\implies$ symmetry, transitivity, and coercion.

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$$\rightsquigarrow \; p : \mathsf{Id}_A(a, b)$$

$$\mathsf{Id}_A(b, a)$$

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$$\mathsf{refl}(a) : \mathsf{Id}_A(a, a) \quad \overset{p \,:\, \mathsf{Id}_A(a,b)}{\rightsquigarrow} \quad \mathsf{Id}_A(b, a)$$

$$p : \mathsf{Id}_A(a, b) \quad \overset{q \,:\, \mathsf{Id}_A(b,c)}{\rightsquigarrow} \quad \mathsf{Id}_A(a, c)$$

Everything respects equality $\Longrightarrow$ symmetry, transitivity, and coercion.

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$$\lambda a.a : A \to A \quad \overset{p \,:\, \mathsf{Id}_{\mathsf{Type}}(A,B)}{\rightsquigarrow} \quad \mathsf{coerce}(p) : A \to B$$

Identity types

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$$\mathrm{refl}(a) : \mathrm{Id}_A(a, a) \;\rightsquigarrow_{p : \mathrm{Id}_A(a, b)}\; \mathrm{Id}_A(b, a)$$

$$p : \mathrm{Id}_A(a, b) \;\rightsquigarrow_{q : \mathrm{Id}_A(b, c)}\; \mathrm{Id}_A(a, c)$$

$$\lambda a.a : A \to A \;\rightsquigarrow_{p : \mathrm{Id}_{\mathrm{Type}}(A, B)}\; \mathrm{coerce}(p) : A \to B$$

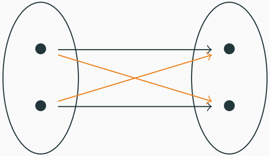
$\not\Longrightarrow$ UIP — there's only one proof of $\mathrm{Id}_A(a, b)$ [Hofmann, Streicher 1998].

Univalence

Univalence axiom [Voevodsky 2010] — for any $f : A \xrightarrow{\sim} B$,

- $\mathsf{univalence}(f) : \mathsf{Id}_{\mathsf{Type}}(A, B)$
- $\mathsf{coerce}(\mathsf{univalence}(f)) : A \to B$ applies f [Licata 2016]

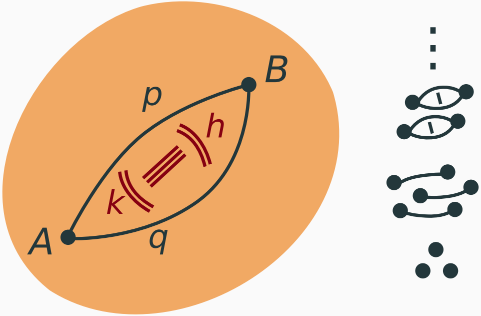
Contradicts UIP — $\mathsf{univalence}(\rightrightarrows) \neq \mathsf{univalence}(\times)$.



Higher-dimensional structure

Contradicts UIPIP, UIPIPIP… [Kraus, Sattler 2015].

This higher-dimensional structure is the same as seen in topology [Voevodsky 2012].

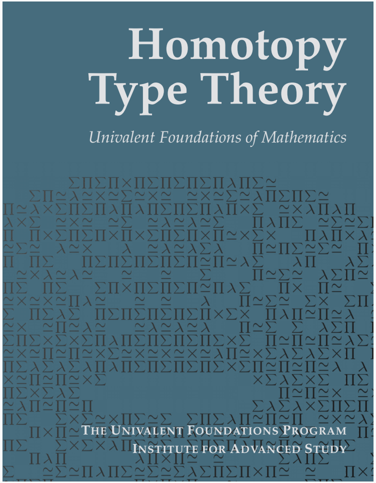


(Image credit: Favonia)

Homotopy type theory

Use type theory + univalence + higher inductive types
to prove theorems about topological spaces!

- Seifert–van Kampen theorem [Favonia, Shulman 2016]
- Blakers–Massey theorem [Favonia *et al.* 2016]
- Gysin sequence [Brunerie 2016]
- Serre spectral sequence [van Doorn 2018]



Computation I

Where do programs enter the picture?

Proof assistants can extract programs from proofs (*e.g.*, CompCert).

		ML Program	**Behavioral Spec.**
$f : (n : \mathbb{N}) \to \mathsf{List}(n)$	$\rightsquigarrow$	$f : \mathbb{N} \to \mathsf{List}$	where $\mathsf{length}(f(n)) = n$
$p : \mathsf{Id}_A(a, b)$	$\rightsquigarrow$		where $a = b$

Univalence $\implies$ p needed at runtime $\implies$ standard extraction is broken.

Computation II

Proof assistants use computation to silently discharge many equations.

$$\mathsf{append} : \mathsf{List}(n) \longrightarrow \mathsf{List}(m) \longrightarrow \mathsf{List}(n+m)$$

$$? : \mathsf{Id}_{\mathsf{List}(2)}(\underbrace{\mathsf{append}\ [\star]\ [\star]}_{:\ \mathsf{List}(1+1)}, [\star, \star])$$

If we can't compute with univalence, proofs using it are quite bureaucratic.

Canonicity

For these reasons, we want a computational semantics for a type theory with univalence — a way to regard proofs involving univalence as programs. Summarized by:

Theorem (Canonicity): Every $\cdot \vdash n : \mathbb{N}$ computes to, and is equal to, a concrete numeral.

Thesis statement

Higher-dimensional types classify higher-dimensional programs extensionally according to their behaviors.

I describe Cartesian cubical type theory ($\times$ 2) and present its computational semantics.

Published at POPL 2017 [A., Harper, Wilson] and CSL 2018 [A., Favonia, Harper].

Implemented in two proof assistants (`http://github.com/RedPRL`):

- **RedPRL** [A., Cavallo, Favonia, Harper, Sterling 2018]
- **`redtt`**

Outline

In the rest of this talk:

- Cartesian cubical type theory
- Computational semantics
- Taking stock

Cartesian cubical type theory

Coercion

$$\mathrm{coercion} : \mathrm{Id}_{\mathsf{Type}}(A, B) \to A \to B$$

Ordinary type theory — at runtime, only $\mathrm{coercion}(\mathrm{refl}(A)) = \lambda a.a : A \to A$.

With univalence — other arguments possible; coercion must do something!

Coercion

Computes by cases on the proof of $\mathrm{Id}_{\mathsf{Type}}(A, B)$. (Not on A, B!)

$$\mathrm{univalence}(f) : \mathrm{Id}_{\mathsf{Type}}(A, B)$$

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$$\mathrm{univalence}(f) : \mathrm{Id}_{\mathrm{Type}}(A, B)$$

$$\mathrm{refl}(A \longrightarrow B) : \mathrm{Id}_{\mathrm{Type}}(A \longrightarrow B, A \longrightarrow B)$$

$$p : \mathrm{Id}_{\mathrm{Type}}(B, B') \rightsquigarrow$$

$$\mathrm{Id}_{\mathrm{Type}}(A \longrightarrow B, A \longrightarrow B')$$

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$$\mathrm{refl}(\mathrm{List}(n)) : \mathrm{Id}_{\mathrm{Type}}(\mathrm{List}(n), \mathrm{List}(n))$$

$$p : \mathrm{Id}_{\mathbb{N}}(n, m) \rightsquigarrow$$

$$\mathrm{Id}_{\mathrm{Type}}(\mathrm{List}(n), \mathrm{List}(m))$$

Coercion

Computes by cases on the proof of $\mathrm{Id}_{\mathrm{Type}}(A, B)$. (Not on A, B!)

$$\mathrm{univalence}(\mathrm{reverse}) : \mathrm{Id}_{\mathrm{Type}}(\mathrm{List}(n), \mathrm{List}(n))$$

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$$p : \mathrm{Id}_{\mathbb{N}}(n, m) \rightsquigarrow$$

$$\mathrm{Id}_{\mathrm{Type}}(\mathrm{List}(n), \mathrm{List}(m))$$

Paths

Where to put the proof that A equals B? In between them!

$$\begin{array}{ccc} A & \xrightarrow{\quad p(x) \quad} & B \\ \| & & \| \\ p(0) & & p(1) \end{array}$$

p is a "continuous function" out of $[0, 1] \subset \mathbb{R}$.

Interval and path types

Cubical type theories add an interval $\mathbb{I}$, path types...

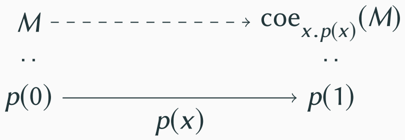
$$\frac{}{\Gamma \vdash 0 : \mathbb{I}} \qquad \frac{}{\Gamma \vdash 1 : \mathbb{I}} \qquad \frac{}{\Gamma, x : \mathbb{I} \vdash x : \mathbb{I}}$$

$$\frac{\Gamma, x : \mathbb{I} \vdash M(x) : A}{\Gamma \vdash \langle x \rangle M(x) : \mathrm{Path}_A(M(0), M(1))} \qquad \frac{\Gamma \vdash M : \overbrace{\mathrm{Path}_A(a_0, a_1)}^{\{p : \mathbb{I} \to A \mid p(0) = a_0 \wedge p(1) = a_1\}} \qquad \Gamma \vdash r : \mathbb{I}}{\Gamma \vdash M\, r : A}$$

Coercion

…and a new coercion operation.

$$\frac{\Gamma, x : \mathbb{I} \vdash p(x) : \mathrm{Type} \qquad \Gamma \vdash M : p(0)}{\Gamma \vdash \mathrm{coe}_{x.p(x)}(M) : p(1)}$$



Coercion

…and a new coercion operation.

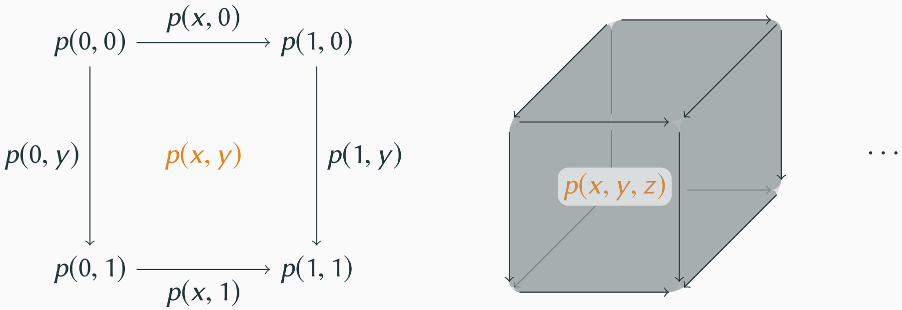
$$\frac{\Gamma \vdash r, r' : \mathbb{I} \qquad \Gamma, x : \mathbb{I} \vdash p(x) : \mathsf{Type} \qquad \Gamma \vdash M : p(r)}{\Gamma \vdash \mathsf{coe}^{r \rightsquigarrow r'}_{x.p(x)}(M) : p(r')}$$

$$= M \quad \text{when } r = r'$$

$$\begin{array}{ccc} M & \dashrightarrow & \mathsf{coe}^{0 \rightsquigarrow 1}_{x.p(x)}(M) \\ \vdots & & \vdots \\ p(0) & \xrightarrow[p(x)]{} & p(1) \end{array}$$

Hypercubes

$\mathbb{I}$ makes it cubical because $p(x_1, \ldots, x_n)$ forms an n-dimensional hypercube.



Cubical models of type theory

	Structure on $\mathbb{I}$		. . . on coercion
BCH [Bezem, Coquand, Huber 2013]	★	*(affine)*	★★
CCHM [Cohen, C., H., Mörtberg 2016]	★ ★ ★	$\min(r, r')$, $\max(r, r')$, $(1 - r)$	★
Cartesian [A., Favonia, Harper 2017]	★★	*(structural)*	★ ★ ★

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Contribution: Model full univalent type theory using structural $\mathbb{I}$.
[Coquand 2014; Brunerie, Licata 2014; Awodey 2016; *et al.*]

Coercion — function types

How does coercion compute? Suppose $p(x) := A(x) \to B(x)$.

(Requires $1 \rightsquigarrow 0$; dependent functions require $1 \rightsquigarrow x$ and $1 \rightsquigarrow 1 = \mathrm{id}_A$.)

$$\frac{x : \mathbb{I} \vdash A(x) : \mathrm{Type} \qquad x : \mathbb{I} \vdash B(x) : \mathrm{Type} \qquad f : A(0) \to B(0)}{\mathrm{coe}^{0 \rightsquigarrow 1}_{x.A(x) \to B(x)}(f) : A(1) \to B(1)}$$

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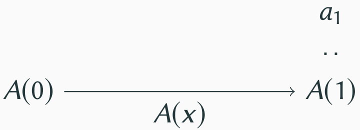

$:= \lambda a_1.$

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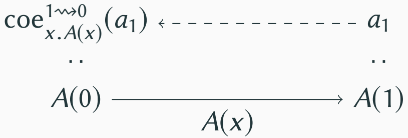

$:= \lambda a_1. \qquad \mathrm{coe}^{1 \rightsquigarrow 0}_{x.A(x)}(a_1)$

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$$:= \lambda a_1. \qquad f\ \mathrm{coe}^{1 \rightsquigarrow 0}_{x.A(x)}(a_1)$$

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$$B(0) \xrightarrow[B(x)]{} B(1)$$

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$$:= \lambda a_1.\mathrm{coe}^{0 \rightsquigarrow 1}_{x.B(x)}(f\ \mathrm{coe}^{1 \rightsquigarrow 0}_{x.A(x)}(a_1))$$

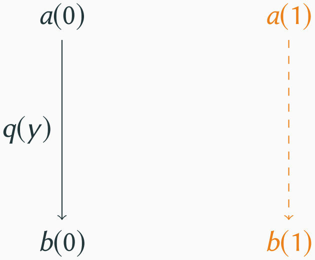
$$f\ \mathrm{coe}^{1 \rightsquigarrow 0}_{x.A(x)}(a_1) \dashrightarrow \mathrm{coe}^{0 \rightsquigarrow 1}_{x.B(x)}(f\ \mathrm{coe}^{1 \rightsquigarrow 0}_{x.A(x)}(a))$$

$$B(0) \xrightarrow[B(x)]{} B(1)$$

Coercion — path types

Suppose $p(x) := \mathrm{Path}_{A(x)}(a(x), b(x))$.

$$\frac{A : \mathrm{Type} \qquad x : \mathbb{I} \vdash a(x) : A \qquad x : \mathbb{I} \vdash b(x) : A \qquad q : \mathrm{Path}_A(a(0), b(0))}{\mathrm{coe}^{0 \rightsquigarrow 1}_{x.\mathrm{Path}_A(a(x), b(x))}(q) : \mathrm{Path}_A(a(1), b(1))}$$



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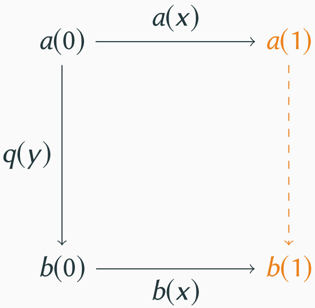
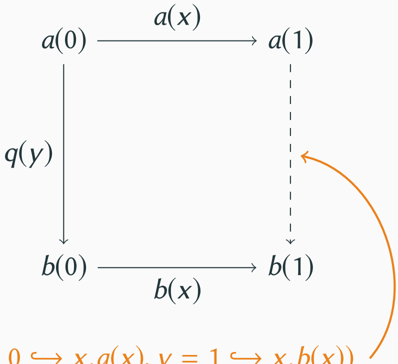


Kan composition

Composition operation extends connected partial cubes to total cubes [Kan 1955].

$$
\frac{
\begin{array}{ll}
 & \Gamma \vdash M : A \\
(\forall i) & \Gamma, \xi_i, x : \mathbb{I} \vdash N_i : A \\
(\forall i, j) & \Gamma, \xi_i, \xi_j, x : \mathbb{I} \vdash N_i = N_j : A \\
(\forall i) & \Gamma, \xi_i \vdash N_i\langle r/x \rangle = M : A
\end{array}
}{
\Gamma \vdash \mathrm{hcom}_A^{r \rightsquigarrow r'}(M; \overrightarrow{\xi_i \hookrightarrow x.N_i}) : A
}
$$

$$
= \begin{cases} M & \text{when } r = r' \\ N_i\langle r'/x \rangle & \text{when } \xi_i \end{cases}
$$



$\mathrm{hcom}_A^{0 \rightsquigarrow 1}(q(y); y = 0 \hookrightarrow x.a(x), y = 1 \hookrightarrow x.b(x))$

“Draw the rest of the owl”

Composition computes by cases on the type, mutually with coercion. Then…

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- Certain base types have free compositions and coercions [Cavallo, Harper 2019]

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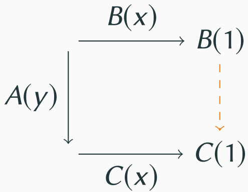
- Need a type former for univalence: $A \xrightarrow{V_x(f \,:\, A \xrightarrow{\sim} B)} B$

"Draw the rest of the owl"

Composition computes by cases on the type, mutually with coercion. Then...

- Certain base types have free compositions and coercions [Cavallo, Harper 2019]
- Need a type former for univalence: $A \xrightarrow{V_x(f \,:\, A \xrightarrow{\sim} B)} B$
- Need a type former for compositions of types:



“Draw the rest of the owl”

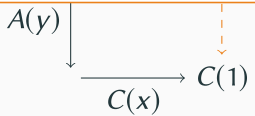
Composition computes

- Certain base types

- Need a type former

$$
\frac{
\begin{gathered}
\widetilde{N}_i[w,z] := \mathsf{coe}^{s'\langle w/x\rangle \rightsquigarrow z}_{z.B_i\langle w/x\rangle}(\mathsf{coe}^{r \rightsquigarrow w}_{x.B_i\langle s'/z\rangle}(M)) \\
\widetilde{T} := \overrightarrow{\xi_i\langle r/x\rangle \hookrightarrow z.\mathsf{coe}^{z \rightsquigarrow s\langle r/x\rangle}_{z.B_i\langle r/x\rangle}(\mathsf{coe}^{s'\langle r/x\rangle \rightsquigarrow z}_{z.B_i\langle r/x\rangle}(M))} \\
\widetilde{O}[z] := \mathsf{hcom}^{s'\langle r/x\rangle \rightsquigarrow z}_{A\langle r/x\rangle}(\mathsf{cap}^{s\langle r/x\rangle \leftsquigarrow s'\langle r/x\rangle}(M; \overrightarrow{\xi_i\langle r/x\rangle \hookrightarrow z.B_i\langle r/x\rangle}); \widetilde{T}) \\
\widetilde{P} := \mathsf{com}^{r \rightsquigarrow r'}_{x.A}(\widetilde{O}[s\langle r/x\rangle]; \overrightarrow{\xi_i \hookrightarrow x.\widetilde{N}_i[x,s]|_{(x\#\xi_i)}}, s = s' \hookrightarrow x.\mathsf{coe}^{r \rightsquigarrow x}_{x.A}(M)|_{(x\#s,s')}) \\
\widetilde{Q}_k[z] := \mathsf{com}^{s\langle r'/x\rangle \rightsquigarrow z}_{z.B_k\langle r'/x\rangle}(\widetilde{P}; \overrightarrow{\xi_i \hookrightarrow z.\widetilde{N}_i[r',z]|_{(x\#\xi_i)}}, r = r' \hookrightarrow z.\mathsf{coe}^{s'\langle r'/x\rangle \rightsquigarrow z}_{z.B_k\langle r'/x\rangle}(M)) \\
\widetilde{H} := \mathsf{hcom}^{s\langle r'/x\rangle \rightsquigarrow s'\langle r'/x\rangle}_{A\langle r'/x\rangle}(\widetilde{P}; \overrightarrow{\xi_i\langle r'/x\rangle \hookrightarrow z.\mathsf{coe}^{z \rightsquigarrow s\langle r'/x\rangle}_{z.B_i\langle r'/x\rangle}(\widetilde{Q}_i[z])}, r = r' \hookrightarrow z.\widetilde{O}[z]) \\
\widetilde{C} := \mathsf{box}^{s\langle r'/x\rangle \rightsquigarrow s'\langle r'/x\rangle}(\widetilde{H}; \overrightarrow{\xi_i\langle r'/x\rangle \hookrightarrow \widetilde{Q}_i[s'\langle r'/x\rangle]})
\end{gathered}
}{
\Psi \mid \Gamma \vdash \mathsf{coe}^{r \rightsquigarrow r'}_{x.\mathsf{hcom}^{s \rightsquigarrow s'}_{\mathsf{Type}_j}(A; \overrightarrow{\xi_i \hookrightarrow z.B_i})}(M) = \widetilde{C} : (\mathsf{hcom}^{s \rightsquigarrow s'}_{\mathsf{Type}_j}(A; \overrightarrow{\xi_i \hookrightarrow z.B_i}))\langle r'/x\rangle
}
$$

- Need a type former for compositions of types:



Computational semantics

Semantics

Dependent type theory + $\mathbb{I}$ + coercion + composition + univalence + $\cdots$ = ?

- Is this consistent?
- Does this give computational meaning to univalence?

Yes and yes — we give a computational semantics [Martin-Löf 1979; Allen 1987].

(Denotational semantics formalized in Agda [A., Brunerie, Coquand, Favonia, Harper, Licata].)

Ordinary computational semantics

$$\cdot \vdash M : A$$

- Interpret every closed proof M as a program

- Define an untyped functional programming language ($M \longmapsto M'$ and M val)
- The interpretation (extraction) can delete annotations

$$\frac{M \longmapsto M'}{M\ N \longmapsto M'\ N} \qquad \frac{}{(\lambda a.M)\ N \longmapsto M[N/a]} \qquad \cdots$$

Ordinary computational semantics

$\cdot \vdash M : A$

- Interpret every closed proof M as a program
- Interpret every closed type A as a behavioral specification

---

- Types are binary ("logical") relations on programs ($M \doteq N \in A$)
- Closed under evaluation — if $M \doteq M \in A$ then $M \Downarrow V$ and $M \doteq V \in A$
- For observable types, return an answer — $M \doteq N \in \text{bool}$ iff $M, N \Downarrow \text{true}$ or $M, N \Downarrow \text{false}$
- Consider only the closed instances of open terms ($a : A \gg M(a) \doteq N(a) \in B(a)$)

Cubical programming language

Cubical programs include coercion/composition.

To case on the path, we must evaluate terms containing interval variables!

$$\frac{M \longmapsto M'}{M\ N \longmapsto M'\ N} \qquad \frac{}{(\lambda a.M)\ N \longmapsto M[N/a]} \qquad \frac{A \longmapsto A'}{\mathrm{coe}^{r \rightsquigarrow r'}_{x.A}(M) \longmapsto \mathrm{coe}^{r \rightsquigarrow r'}_{x.A'}(M)}$$

$$\frac{}{\mathrm{coe}^{0 \rightsquigarrow 1}_{x.A(x) \to B(x)}(M) \longmapsto \lambda a_1.\mathrm{coe}^{0 \rightsquigarrow 1}_{x.B(x)}(f\ \mathrm{coe}^{1 \rightsquigarrow 0}_{x.A(x)}(a_1))}$$

$$\frac{}{\mathrm{coe}^{0 \rightsquigarrow 1}_{x.\mathrm{Path}_A(a(x), b(x))}(q) \longmapsto \mathrm{hcom}^{0 \rightsquigarrow 1}_{A}(q(y); y = 0 \hookrightarrow x.a(x), y = 1 \hookrightarrow x.b(x))} \qquad \dots$$

Cubical logical relations

Cubical behavioral specifications range over programs with interval variables.

$$M \doteq N \in A\,[x_1, \ldots, x_n]$$

Cubical logical relations

Cubical behavioral specifications range over programs with interval variables.

$$\underbrace{a_1 : A_1, \ldots, a_n : A_n}_{\text{(extensionally)}} \gg M \doteq N \in A\,[x_1, \ldots, x_n]$$

Coherence

These specifications must be closed under evaluation and interval substitutions.

$$M \doteq M \in A\,[\Psi] \implies M \Downarrow V \text{ and } M \doteq V \in A\,[\Psi]$$
$$M(x) \doteq N(x) \in A\,[\Psi, x] \implies M(0) \doteq N(0) \in A\,[\Psi]$$

Thus,

$$M(x) \doteq M(x) \in A\,[\Psi, x] \implies \begin{array}{c} M(x) \doteq V(x) \in A\,[\Psi, x] \\ M(0) \doteq V(0) \in A\,[\Psi] \\ M(0) \doteq V' \in A\,[\Psi] \end{array}$$

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Proving coherence

Much of the difficulty involves proving coherence of evaluation and interval substitution.

- Ordinary steps commute on the nose with interval substitution. (Cubically stable.)
- To show these commute for M, construct a substitution-indexed family of its reducts, and prove they are pairwise extensionally equal. (Coherent expansion.)

Canonicity

Booleans (and HITs!) are indeed observable.

Theorem: Every $M \in \text{bool}\ [\cdot]$ computes to, and is equal to, true or false. ($\approx$ [Huber 2016])

Theorem: Every $M \in \mathbb{S}^1\ [\cdot]$ computes to, and is equal to, base.

Contribution: Refinement of composition operation ("validity" restriction) makes higher inductive types also observable. [Vezzosi, Mörtberg, Abel 2019]

Exact equality

These specifications naturally account for extensional/exact equality $M \doteq N \in A\ [\cdot]$.
(Unlike paths, these equations do not appear in extracts.)

Can we have an exact equality type? No, it doesn't support coercion!

$$\star \in \mathsf{Eq}_{\mathsf{Type}}(A, A)\ [\cdot]$$

$$\rightsquigarrow \ \mathsf{univalence}(f : A \xrightarrow{\sim} B)$$

$$\mathsf{Eq}_{\mathsf{Type}}(A, B)\ [\cdot] \ ✗$$

Two-level type theory

We present a two-level type theory [Voevodsky 2013; Altenkirch, Capriotti, Kraus 2016]:

- Pretypes — respect only exact equality
- Kan types — respect paths and exact equality

Exact equality is seemingly needed for some mathematical constructions.

Implementing exact equality

Practical question — what principles about exact equality do we expose?

“Equality reflection” precludes type-checking proofs (as in Coq, Agda, **redtt**, …).

Nuprl/**RedPRL** include reflection; instead of type-checking, one directly manipulates (cubical) extracts and behavioral specifications as a program logic.

Contribution: **RedPRL** is the first (only) implementation of a two-level type theory with canonicity.

Taking stock

Contribution

We present Cartesian cubical type theory, a univalent dependent type theory whose proofs have computational meaning.

Second such, after the cubical type theory of [Cohen, Coquand, Huber, Mörtberg 2016].

Try these out in **RedPRL**, **redtt**, and Cubical Agda!

What's all this good for? I

First answer — the homotopy type theory project has led mathematicians to use type theory as a formal language for homotopy theory.

In our experience, type theories with good computational properties are easier to use.

Serves as a language for topological spaces — equivariant Cartesian cubical model [Awodey, Cavallo, Coquand, Riehl, Sattler].

What's all this good for? II

Second answer — cubes solve some longstanding difficulties with equality connectives.

- Function extensionality, unlike identity types
- Good account of dependent equality ("pathovers")
- Well-behaved quotients (better with observable HITs!)
- Univalence permits invariant view of mathematics [Awodey 2014]

We've backported cubes to non-univalent type theory — XTT [Sterling, A., Gratzer 2019].

Thanks!

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- Chapters 1 and 5 — Big picture
- Chapter 2 — Ordinary computational semantics
- Chapter 3 — Cubical type theories
- Chapter 4 — All main theorems
- Appendix A — Program logic ≈ **RedPRL**
- Appendix B — **`redtt`** core calculus

Coercion — dependent function types

Suppose $p(x) := (a : A(x)) \to B(x, a)$. (Uses $1 \rightsquigarrow x$ and $1 \rightsquigarrow 1$.)

$$\frac{x : \mathbb{I} \vdash A(x) : \mathrm{Type} \qquad x : \mathbb{I}, a : A \vdash B(x, a) : \mathrm{Type} \qquad f : (a_0 : A(0)) \to B(0, a_0)}{\mathrm{coe}^{0 \rightsquigarrow 1}_{x.A(x) \to B(x,a)}(f) : (a_1 : A(1)) \to B(1, a_1)}$$

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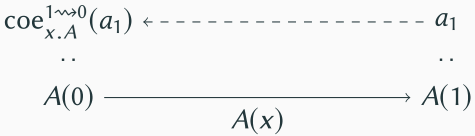

$$:= \lambda a_1. \qquad \mathrm{coe}^{1 \rightsquigarrow 0}_{x.A(x)}(a_1)$$

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$$:= \lambda a_1. \qquad f\ \mathrm{coe}^{1 \rightsquigarrow 0}_{x.A(x)}(a_1)$$

$$f\ \mathrm{coe}^{1 \rightsquigarrow 0}_{x.A(x)}(a_1) \dashrightarrow$$

$$B(0, \mathrm{coe}^{1 \rightsquigarrow 0}_{x.A(x)}(a_1)) \qquad B(1, a_1)$$

Coercion — dependent function types

Suppose $p(x) := (a : A(x)) \to B(x, a)$. (Uses $1 \rightsquigarrow x$ and $1 \rightsquigarrow 1$.)

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$$:= \lambda a_1.\mathsf{coe}^{0 \rightsquigarrow 1}_{x.B(x, \mathsf{coe}^{1 \rightsquigarrow x}_{x.A}(a_1))}(f\ \mathsf{coe}^{1 \rightsquigarrow 0}_{x.A(x)}(a_1))$$

$$\begin{array}{ccc} f\ \mathsf{coe}^{1 \rightsquigarrow 0}_{x.A(x)}(a_1) & \dashrightarrow & \\ \vdots & & \vdots \\ B(0, \mathsf{coe}^{1 \rightsquigarrow 0}_{x.A(x)}(a_1)) & \xrightarrow[B(x, \mathsf{coe}^{1 \rightsquigarrow x}_{x.A}(a_1))]{} & B(1, a_1) \\ & & \parallel \\ & & \mathsf{coe}^{1 \rightsquigarrow 1}_{x.A}(a_1) \end{array}$$

Graphics test

Lorem ipsum dolor sit amet, consectetur adipiscing elit, sed do eiusmod tempor incididunt [ut labore et dolore magna aliqua]. Utenimadminimveniam, **RedPRL**

quis nostrud: exercitation ullamco laboris nisi ut aliquip ex ea commodo [consequat].

Duis aute irure dolor in reprehenderit in voluptate velit esse cillum *dolore* ★★