JOINTLY PERIODIC POINTS IN CELLULAR AUTOMATA

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Abstract. For a one-dimensional surjective cellular automaton, we consider temporal periodicity of spatially periodic points. We offer some questions, facts and experimental results.

Contents

1. Introduction

In this paper we consider the action of a surjective one-dimensional cellular automaton map f on spatially periodic points. (In the language of symbolic dynamics [9, 11, 13]), f is a surjective endomorphism of S_N, the full shift on N symbols, where N is the size of the symbol set of the automaton.) This paper is primarily an experimental mathematics paper, based on data from a computer program written by author Lee. The experimental results and mathematical context, which we explain, lead us to questions and a conjecture on the growth rate of the spatially periodic points which are also temporally periodic points.

To express the questions and conjecture, we make the following definitions. Let $P_k(S)$ denote the points of period k of S, i.e. the points fixed by S^k, and let $\mathrm{Per}(S) = \cup_k P_k(S)$. So, $\mathrm{Per}(S_N)$ is the set of spatially periodic points for a one-dimensional cellular automaton on N symbols. Let $\nu_k(f, S_N)$ denote the number of jointly periodic points in $P_k(S)$ (i.e. points periodic under f as well as S_N), and define

$$\nu(f, S_N) = \limsup_k \nu_k(f, S_N)^{1/k} \ .$$

These definitions make sense for any f commuting with any $S_N : X \to X$.

Question 1.1. *Is it true for every surjective one-dimensional cellular automaton f on N symbols that $\nu(f, S_N) \geq \sqrt{N}$?*

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Question 1.2. *Is it true for every surjective one-dimensional cellular automaton f on N symbols that $\nu(f, S_N) > 1$?*

Conjecture 1.3. *There exists $N > 1$ and a surjective cellular automaton f on N symbols such that $\nu(f, S_N) < N$*

In Section 2, we give detailed definitions and background. In Section 3, we describe some mechanisms by which one can prove lower bounds for $\nu(f, S_N)$ for some f, and discuss the questions. In Section 4, we discuss the experimental evidence (which is collected in tables at the end). In Section 5, we discuss the computer program we use. We collect the data tables in Section 6.

2. Definitions and background

In this section we recall some background. For a thorough introduction to these topics, see [11] or [13].

Let $\mathcal{A}$ be a finite set of N elements (the symbol set), with the discrete topology. Let Σ_N be the product space $\mathcal{A}^{\mathbb{Z}}$, with the product topology. We view a point x in Σ_n as a doubly infinite sequence of symbols from S, so $x = \dots x_{-1}x_0x_1\dots$. The space Σ_N is compact metrizable, and one metric compatible with the topology is $\text{dist}(x, y) = 1/(|n| + 1)$ where $|n|$ is the minimum nonnegative integer such that $x_n \neq y_n$. A set E is *dense* in Σ_N in this topology if for every k and every word W in $\mathcal{A}^{2k+1}$ there exists x in E such that $x[-k, k] = W$.

The shift map $\sigma : \Sigma_n \to \Sigma_n$ is the homeomorphism defined by the rule $(\sigma x)_i = x_{i+1}$. The topological dynamical system (Σ_N, σ) is called the *full shift on n symbols*. If X is a closed shift-invariant subsystem of some full shift, then $(X, \sigma|_X)$ is a *subshift*. If X is exactly the subsystem of points which never see some given finite set of words, then $(X, \sigma|_X)$ is a *shift of finite type*. These subshifts are useful and natural for the study of cellular automata (see the remarks in [4]). For a lighter notation, we may sometimes use a single letter like S to represent a subshift $(X, \sigma|_X)$ or the restricted shift map $\sigma|_X$.

The *entropy* $h(S)$ of a subshift $S = (X, \sigma|_X)$ is the growth rate of S-words: $h(S) = \lim_k (1/k) \log(|\{x[1, k] : x \in X|\})$. In particular, $h(S_N) = \log N$.

A map $f : \Sigma_N \to \Sigma_N$ is a continuous and shift-commuting ($f\sigma = \sigma f$) if and only f is a *block code*, i.e. there exist integers a, b and a function $F : \mathcal{A}^{b-a+1} \to \mathcal{A}$ such that $(fx)_i = F(x[i + a, i + b])$ for all integers i and all $x \in \Sigma_N$. Such a map f is called a one-dimensional cellular automaton map. If $b-a+1$ is minimal for f, then it is the *span* of f. There is a well known dichotomy for such maps f: either (i) f is surjective and for some integer M every point has at most M preimages, or (ii) image points typically have uncountably many preimages, and f is not surjective [9, 11, 13]. For an onto c.a. map f, there is a positive integer d, the *degree* of f, such that most points have d preimage points under f, and no point has fewer than d preimage points.

We restrict our attention to surjective maps in this paper because we are interested in periodic points of f, which must be contained in $\cap_{k>0} f^k \Sigma_N$, the eventual image of f. We separate our ignorance about periodic points from additional difficulties involving the passage to the eventual image [14].

If N is prime, then every c.a. map f on N symbols can be presented by a unique polynomial [9] over the finite field $\mathbb{Z}/N$. For example, $x_{-1} + x_1x_2$ defines the c.a. map f such that for all x, $(fx)_i = x_{i-1} + x_{i+1}x_{i+2}$. (If N is not prime, there is still a representation by products of polynomials over suitable finite fields [15], which we won't need.) If $(fx)_0$ is determined by $x[a,b]$, with a maximal and b minimal, then $b-a+1$ is the *span* of f. If a is not specified, then it is assumed $a=0$. A c.a. map on S_N can also be defined by a *lookup code*, as in tables 8 and 9. For example, with $N=2$, order the 2^3 words on symbols $\{0,1\}$ lexicographically (000,001,010,011,100,101,110,111); then the lookup code for a span 3 c.a. f is given by the word $c_1c_2\dots c_8$ on $\{0,1\}$ such that $(fx)_0=c_i$ if $x_0x_1x_2$ is the ith word in lexicographic order. E.g., $x_0+x_1x_2$ corresponds to lookup code 00011110.

A block code $f:\Sigma_N\to\Sigma_N$ is *right-closing* if it never collapses distinct left-asymptotic points. This means that if $f(x)=f(x')$ and for some I it holds that $x_i=x'_i$ for $-\infty<i\le I$, then $x=x'$. The definition of *left closing* is given by reversing the inequality. The map f is *closing* if it is either left or right closing. A closing c.a. map is surjective. An endomorphism of a full shift S_N is constant-to-one if and only if it is both right and left closing.

If f is given by a polynomial which is linear in the leftmost variable (e.g. $x-1+x_1x_2$), then f is left closing. If a map is linear in the leftmost variable x_i, with $i=0$, then f is called left permutative. If f is linear in the rightmost variable, then f is called right permutative, and f is right closing.

Closing maps are important in the coding theory of symbolic dynamics [11, 13]. The immediate relation to the current paper is the following fact: if f is a closing one-dimensional cellular automaton map, then the jointly periodic points in the cellular automaton are dense [4]. It is an open question as to whether every surjective cellular automaton map has dense periodic points. This is a good example of the difficulty of resolving the most basic questions about periodic points and cellular automata.

There are many onto c.a. maps. Let a_k,b_k,c_k be the number of span k c.a. maps on S_N which are respectively injective, surjective, arbitrary (so, $c_k=N^{N^k}$). If d_k is any of the quantities $a_k,b_k,c_k,c_k/b_k,b_k/a_k$, then $\lim_k(1/k)\log\log d_k=\log N$ (for $d_k=a_k$ this was proved in [10]; quantitative estimates related to the rest were to our knowledge first written in an unpublished manuscript of Milnor FIND REFERENCE. There are more constructive abundance results: many automorphisms of subsystems extend to automorphisms of S_N [6], and closing maps of subsystems extend to closing maps of S_N [1].

Given a map g, we let $\mathrm{Fix}(g)$ or $P_k(g)$ denote the set of fixed points of g^k. E.g., $\mathrm{Fix}(S_N)=N^k$. We let $P^o_k(g)$ denote the points of least period k of g. E.g., $P^o_3(S_2)=8-2=6$. $P_k(S_N)$ is mapped into itself by any c.a. f. We define

$$\nu_k(f,S_N)=|\{x\in P_k(S_N):\exists j>0, f^jx=x\}|\ ,\quad \nu(f,S_N)=\limsup_k \nu_k(f,S_N)^{1/k}\ ,$$

$$\nu^o_k(f,S_N)=|\{x\in P^o_k(S_N):\exists j>0, f^jx=x\}|\ ,\quad \nu(f,S_N)=\limsup_k \nu^o_k(f,S_N)^{1/k}\ .$$

For a c.a. map f of S_N, the *jointly periodic* points are those which are both f-periodic (temporally periodic) and S_N-periodic (spatially periodic). We record a remark indicating the subtlety of the periodic behavior of the jointly periodic points.

Remark 2.1. Suppose ϕ and ψ are automorphisms of S_N and N is prime. Possibly after replacing ϕ,ψ with $\phi S^j,\psi S^\ell$ for some integers j,ℓ, there will exist a $\kappa>0$ such that

$$|\mathrm{Fix}(S^m\phi^n)|=|\mathrm{Fix}(S^m\psi^n)|=|\mathrm{Fix}(S^m)|$$

whenever $0 < |m/n| < \kappa$ [5]. At the same time, the sequences $(|\mathrm{Fix}(\phi^n)|)$ and $(|\mathrm{Fix}(\psi^n)|)$ can be very different. A dramatic example of this sort in the setting of shifts of finite type is part of [16, Example 10.1].

Remark 2.2. Suppose $x \in \mathrm{Per}(S_N)$ and f is an endomorphism of $S_N = S$. Then x is in $\mathrm{Per}(f)$ if and only if for some $i > 0$, $f^i x$ and x are in the same S-orbit. Thus for all integers i, j, k, m with m, i, k positive, we have $\nu_k(f, S) = \nu_k(f^i S^j, S^m)$, and consequently $\nu(f, S) = \nu(f^i S^j, S^m)$.

3. Some mechanisms for periodicity

Throughout this section f denotes a c.a. map on N symbols. In this section, we discuss four ways to demonstrate $\nu(f, S_N)$ is large:

(1) find a large shift fixed by f (or more generally by a power of f)
(2) let f be a group endomorphism
(3) use the algebra of a polynomial presenting f
(4) find equicontinuity points.

After discussing these, we offer a random maps heuristic and a question.

We will exhibit the first mechanism in some generality. We prove Proposition 3.2 below to show that there is no property of f considered just as a quotient map (i.e., a map between systems which is not iterated) which prevents $\nu(f, S_N)$ from being arbitrarily close to N. To avoid a digression into detailed symbolic dynamics, we only outline the proof of the technical lemma we need for this result. Below, e.g. $f\phi$ denotes the composition $f \circ \phi$, i.e. ϕ followed by f.

Lemma 3.1. *Suppose S is a mixing SFT, and $f : S \to S$ is a surjective block code, and $\epsilon > 0$. Then there is an automorphism ϕ of (X_A, S) and a shift of finite type T contained in S such that $h(T) > h(S) - \epsilon$ and the fixed point set of ϕf contains T.*

Outline of proof. First, using magic words [11, 13] construct an SFT T_1 in S such that $h(T_1) > h(S) - \epsilon/2$ and the restriction of f to T_1 is injective. Then pick M such that for every $n \geq M$, S has at least two orbits of length n which are not in T_1. Then choose T to be an SFT inside T_1 such that T has no orbits of length smaller than M and also such that $h(T) > h(S) - \epsilon$. Now the image of T is an SFT T' isomorphic to T, and there is a block code $g : T' \to T$ such that on T, gf is the identity map. By [6, Theorem 1.5], there is an automorphism ϕ of S whose restriction to T' equals g. Clearly the restriction of ϕf to T is the identity map. □

Proposition 3.2. *Suppose f is a surjective c.a. on N symbols and $\epsilon > 0$. Then there is an invertible c.a. ϕ such that* $\log(\nu(\phi f, S_N) > h(S_N) - \epsilon$.

Proof. If T is a shift of finite type, then $\limsup_k (1/k) \log |\mathrm{Fix}(T^k)| = h(T)$. Given f, choose ϕ and T as in Lemma 3.1, then $\nu_k(f\phi, S_N) \geq |\mathrm{Fix}(T^k)|$. Consequently $\log(\nu(\phi f), S_N)) \geq \log(N) - \epsilon = h(S_N) - \epsilon$. □

Remark 3.3. The statements of Lemma 3.1 and Proposition 3.2 remain true if ϕf is replaced by $f\phi$. One way to see this is to notice that the systems $(f\phi, S)$ and $(\phi(f\phi)\phi^{-1}, \phi S \phi^{-1}) = (\phi f, S)$ are isomorphic.

Now we turn to algebra. The points of S_N form a group under coordinatewise addition (mod N), and some c.a. are endomorphisms of this group; in the terminology of [15], these c.a. are *linear*. Such maps are given by nonconstant polynomials which are sums of monomials.

Proposition 3.4. *Suppose a surjective c.a. map f is linear on S_N. Then for all large primes p, $\nu_p(f, S_N) \geq N^{p-1}$. Therefore $\nu(f, S_N) = N$.*

Proof. We use an argument from the proof of a related result [4, Proposition 3.2]. Let M be the cardinality of the kernel of f. Suppose $p > M$ and p is prime; then f must map orbits of length p to orbits of length p (otherwise, some orbit of length p would be collapsed to an orbit of length dividing p, i.e. to a fixed point, which would contradict the fact that every point has M preimages, with $M < p$). Therefore, for all k, the kernel of f^k contains no point in an orbit of length p. The fixed points of $(S_N)^p$ form a subgroup H which is mapped into itself by f, so for some $k > 0$, the restriction of f^k to $f^k H$ is injective, and all points in the set $f^k H$ are f-periodic. Because the kernel of $(f^k)|H$ contains at most N points, it follows that at least $1/N$ of the fixed points of S^p are periodic for f. □

In the linear case, the algebraic structure allows a number theoretic description of the way that f-periods of jointly periodic points of S_N period k vary (quite irregularly) with k [15]. In the nonlinear case we have nothing close to such an understanding.

Algebra can be used in a different way. Again, cellular automata maps can be presented by polynomials [9, 15]. Frank Rhodes [18] used algebraic properties of these to exhibit a family of one-dimensional noninjective c.a. such that $\nu_k^o(f, S_N) = |P_k^o(S_N)|$ for all k outside $m\mathbb{N}$, where m is a positive integer greater than 1. Clearly in this case $\nu(f, S_N) = N$. We will not review that argument.

We now turn to equicontinuity. A point x is equicontinuous for f if there are positive integers K, M such that for all points y, if $x[-K, K] = y[-K, K]$ then for all $n > 0$ $(f^n x)[-M, M] = (f^n y)[-M, M]$. If the surjective c.a. f has a point of equicontinuity, then $\lim_n (1/n) \log \nu_n(f, S_N) = \log(N)$ [2]. Points of equicontinuity occur in various natural examples (e.g. [3, 12]).

For most surjective c.a., the criteria above are not applicable. This leads to the experimental investigations discussed in the next section, and to the possibility raised in Questions 1.1 and 1.2 of a general plenitude of spatially periodic points. Question 1.1 arises because in the experimental data, the restrictions of the c.a. f to $P_k(S_N)$ are somewhat reminiscent of a random map on a finite set. Since f is a surjective c.a. map, there is an M such that no point has more than M preimages under f. Suppose for example k is a prime greater than M and let $\mathcal{O}_k(S_N)$ denote the set of S_n orbits of size N. Then f defines an at most M-to-1 map f_k from $\mathcal{O}_k(S_N)$ into itself, and it is perhaps a plausible heuristic that, in the absence of some additional structure, members of the sequence (f_k) will display some of the behavior of random maps; and that "additional structure" such as existence of equicontinuity points may serve to put a floor rather than a ceiling on the numbers $\nu_k(f, S_N)$. The beautiful and extensive theory of random maps on finite sets provides precise asymptotic distributions answering various natural questions [19]. Here we simply note that for a random map on a set of K elements: asymptotically on the order of $\sqrt{K}$ of the elements will lie in cycles (whether the map is bounded-to-one as in [7, Theorem 2], or not [19]); the longest cycle will be on the order of $\sqrt{K}$; and there will be few big cycles.

The maps f_k, derived from the surjective c.a. f, do have one other obvious nonrandom property [9, 11, 13]: most points have the minimal possible number of preimages. This does not work against the heuristic behind Question 1.1.

We cannot answer Question 1.2 even in the case where f is a "closing" map and we know there is an abundance of jointly periodic points [4]. Conjecture 1.3 is a stronger expression of our ignorance. From the experimental data discussed below, it seems clear that there will be many surjective c.a. f with $\nu(f, S_N) < N$. However, we cannot give a proof for any example.

4. Experimental results

With two exceptions, f below is a surjective c.a. defined on the full shift S_N with $N = 2$, with symbol set $\{0, 1\}$. The two exceptions are Tables 18 and 19, where $N = 3$. In the tables which display individual maps, in row k counts are made from the N^k points of (not necessarily least) shift period k. P denotes the number of points of which are also periodic under f, and L denotes the number of points in the largest f-cycle. The preperiod of a point x is the smallest nonnegative integer j such that $f^j(x)$ is f-periodic. The period of a point is the size of the f-cycle into which it moves under the action of f. Note that the data for $P^{1/k}$ and $L^{1/k}$ is perhaps suggestive of a common limsup of at least $\sqrt{N}$.

The computer program gives more information than this; it will list the lengths and multiplicities of all f-cycles, and all f-preperiods. We have suppressed most of this (which can be found at Boyle's web site) for reasons of space.

Example 4.1. [A linear map] In Table 1, we exhibit the output for a familiar example, the map f defined by the rule $f(x)[0] = x[0] + x[1]$ (where addition is mod 2). This map is linear (a group endomorphism) and everywhere 2-to-1.

Example 4.2. [Permutative] In Table 2, we exhibit results for $x_0 + x_1x_2$, a degree 1 map which is not right closing but which is left permutative and hence left closing.

Example 4.3. [Not closing] In Table 3, we consider a map which is neither left closing nor right closing. The map is constructed by composing a not-left-closing map and a not-right-closing map.

Example 4.4. [2-to-1 linear composed with degree 1 closing] In Table 4, the c.a. map is a composition, the linear degree two map $x_0 + x_1$ followed by a degree one left permutative (hence left closing) map.

Example 4.5. [Linear composed with automorphism] In Table 5, we exhibit results for a biclosing map which is not linear in the end variables. For this, we simply compose the map of Example 4.1 with an involution (an automorphism U such that U^2 is the identity map). The immediately striking feature is that this composition drastically reduces the sequence (ν_n) in comparison with Example 4.1.

Example 4.6. [Bipermutative] In Table 6, we exhibit results for the nonlinear map $x_{-1} + x_0x_1 + x_2$, which is however linear in both end variables. The results don't seem all that different from the biclosing example 4.5 above.

Example 4.7. [Closing] In Table 7, we exhibit results for a left closing map which is not right closing and is not linear in the leftmost variable. This map is the composition of the left permutative map of Table 2 with an involution.

Hedlund, Appel and Welch conducted an early investigation [8] in which they found all surjective c.a. on two symbols of span at most five. (This was not trivial, especially in 1963, because there are 2^{32} c.a. maps of span at most five.) Among all onto maps of span at most four, there are exactly 32 which are not linear in an end variable and which send the point $\dots 0000 \dots$ to itself. There are listed in Table 8. Any other span four onto map which is not linear in an end variable is one of these 32 maps g precomposed or postcomposed with the flip map F (given by the polynomial $x_0 + 1$). Because $gF = F(Fg)F = F^{-1}(Fg)F$, the jointly periodic data for Fg and gF will be the same. We exhibit $\nu_k^o(\cdot, S_2)$ in Tables 10 – 13 for the 32 maps g and the 32 maps Fg. ($\nu_k^o(\cdot, S_2)$ counts out of points of least shift period k while $\nu_k(\cdot, S_2)$ counts out of points of (not necessarily least) shift period k.)

According to [8], there are 141,792 surjective c.a. maps of span 5. These are arranged in [8] into classes – linear in end variables, compositions of lower-span maps, remainder. The remainder class (11,388 maps) is broken down into subclasses by patterns of generation, and a less regular residual class of 200 maps. The residual class is generated with various operations by the 26 maps we copy in Table 9 from [8, Table XII]. As some kind of sample, in Tables 14 and 15 we display $\nu_k(\cdot, S_2)$ ($k \leq 19$) for these 26 maps of span 5.

In Table 16, we generate a sample of 16 span 5 resolving maps as follows. Let $p_n(x[0,4])$ denote the polynomial rule map for map n in Table 16 and let $q_n(x_0, x_1, x_2, x_3)$ denote the polynomial rule for the map n in Table 8. Then p_n is defined by $p_n = x_0 + q_n(x_1, x_2, x_3, x_4)$. The purpose of Table 16 is to make a rough comparison of a sample of maps which are and are not linear in an end variable. We see no particular difference.

In Table 17 we give a composition of an arbitrarily chosen pair of span 4 onto maps. The decay in ν_k with k is similar to to that in the previous tables.

Finally, in Tables 18 and 19 we look at two examples on the 3-shift. We can't for computer memory reasons look at orbits of long shift period.

5. Discussion of the computer program

Our data come from the program FPeriod developed by author Lee. FPeriod takes as input N, the number of symbols in the shift; k, the shift period of the points; and F, the function that induces the block code f. It tests all points that are shift periodic of period k and determines how many are f-periodic, as well as listing the multiplicity of points for each f-period, the multiplicity of orbits and their periods, and the multiplicity of preperiods for the non-periodic points.

The algorithm. The points x of (not necessarily least) period k are in bijective correspondence with the words $x[0, k-1]$. The program computes $(fx)[0, k-1]$ from $x[0, k-1]$ using the rule F. The main idea of the underlying algorithm is to start with a word of length k and continuously apply f, storing the generated points in a list, until some point is encountered twice. Then, the period and preperiod of all the points in the list are known. Store all the points from the list, with their period and preperiod information, in a table. Then, starting from the next point not in the table, continuously apply f and store the generated points until one of two conditions is met. Either some point is encountered twice, allowing the periods and preperiods for the new points to be calculated, or some point already in the table is encountered, in which case that point's period and preperiod information can likewise be used to calculate the periods and preperiods for the new points. Continue until the period and preperiod for all points of period k are found.

Algorithm complexity. The number of add-to-table, find-in-table, and evaluate-f operations is proportional to N^k. Thus, the time required by the algorithm is proportional to N^k. The add-to-table and find-in-table operations take time proportional to a small constant, so the time taken by the evaluate-f operations dominates the time of the overall algorithm. The memory required by the algorithm is also proportional to N^k because N^k points are eventually stored in the table.

The FPeriod program. FPeriod is a program written in the C++ programming language that implements the algorithm given above. (An earlier Java version ran much more slowly.) It runs under Unix and related operating systems. It and its source code are freely available under the GNU General Public License and can be downloaded at http://www.math.umd.edu/∼mmb/. The more detailed output from which the tables of the current paper were taken are also available there.

The program uses exponential amounts of time and memory, as per the algorithm, but in practice memory is the limiting factor. The program is quite fast because the f-evaluation operation is done by lookup tables. Although the user may input the polynomial that induces f as a formula, the formula is converted into a table so it is faster to evaluate. Functions are only evaluated through actual computation if they are too large to store in a table.

The amount of memory used by the program limits the size of N and k that can be used. By nature of the algorithm, storing N^k points cannot be avoided because it is necessary to remember which points were previously processed. Also, the storage of period and preperiod information for previously processed points is what allows new period and preperiod information to be derived quickly.

FPeriod uses 16 bytes of memory for each point stored in the table. An additional several bytes per point are used by the table itself and by the dynamic memory allocation system. In practice, running the program using $N = 2$ and $k = 26$ required 1.8 gigabytes (1.8 billion bytes) of memory, close to the limit of a high-end computer. Note that this is more than (16 bytes) x (2^{26}) = 1.1 gigabytes and less than (32 bytes) x (2^{26}) = 2.1 gigabytes, which was reasonable to expect. Exponential memory use is inherent in the algorithm, and it is not feasible to improve the memory use of the FPeriod program or a similar program enough to allow exploring significantly larger N or k values.

Additional features of the program include the ability to induce block codes from compositions of functions; the ability to separately track period and preperiod data for points of least period k; and a truncated version which produces just output for ν_k^o.

The FPeriod program was originally developed to produce experimental evidence relevant to the open question of whether a surjective c.a. map must have dense periodic points.

6. Tables

k	Fraction Periodic	kth root of P	kth root of L	P	L	Not-P	Average Period	Average Preperiod	Maximum Preperiod
1	0.500000	1.00	1.00	1	1	1	1.00	0.50	1
2	0.250000	1.00	1.00	1	1	3	1.00	1.25	2
3	0.500000	1.58	1.44	4	3	4	2.50	0.50	1
4	0.062500	1.00	1.00	1	1	15	1.00	3.06	4
5	0.500000	1.74	1.71	16	15	16	14.12	0.50	1
6	0.250000	1.58	1.34	16	6	48	5.12	1.25	2
7	0.500000	1.81	1.32	64	7	64	6.91	0.50	1
8	0.003906	1.00	1.00	1	1	255	1.00	7.00	8
9	0.500000	1.85	1.58	256	63	256	62.05	0.50	1
10	0.250000	1.74	1.40	256	30	768	29.01	1.25	2
11	0.500000	1.87	1.69	1,024	341	1024	340.67	0.50	1
12	0.062500	1.58	1.23	256	12	3840	11.57	3.06	4
13	0.500000	1.89	1.67	4,096	819	4096	818.80	0.50	1
14	0.250000	1.81	1.20	4,096	14	12,288	13.89	1.25	2
15	0.500000	1.90	1.19	16,384	15	16,384	14.99	0.50	1
16	0.000015	1.00	1.00	1	1	65535	1.00	15.00	16
17	0.500000	1.92	1.38	65,536	255	65,536	254.33	0.50	1
18	0.250000	1.85	1.30	65,536	126	196,608	125.73	1.25	2
19	0.500000	1.92	1.62	262,144	9,709	262,144	9708.96	0.50	1
20	0.062500	1.74	1.22	65,536	60	983,040	59.88	3.06	4
21	0.500000	1.93	1.21	1,048,576	63	1,048,576	62.99	0.50	1
22	0.250000	1.87	1.34	1,048,576	682	3,145,728	681.67	1.25	2
23	0.500000	1.94	1.39	4,194,304	2,047	4,194,304	2047.00	0.50	1

TABLE 1. $x_0 + x_1$ on the 2-shift: an algebraic, two-to-one map.

k	Fraction Periodic	kth root of P	kth root of L	P	L	Not-P	Average Period	Average Preperiod	Maximum Preperiod
1	0.500000	1.00	1.00	1	1	1	1.00	0.50	1.00
2	0.750000	1.73	1.00	3	1	1	1.00	0.25	1.00
3	0.500000	1.58	1.00	4	1	4	1.00	0.88	2.00
4	0.687500	1.82	1.41	11	4	5	2.50	0.31	1.00
5	0.812500	1.91	1.71	26	15	6	9.75	0.19	1.00
6	0.281250	1.61	1.00	18	1	46	1.00	3.95	8.00
7	0.609375	1.86	1.74	78	49	50	37.75	0.94	5.00
8	0.667969	1.90	1.81	171	120	85	75.94	0.86	6.00
9	0.482422	1.84	1.55	247	54	265	44.93	2.83	12.00
10	0.535156	1.87	1.82	548	410	476	345.04	2.85	17.00
11	0.183105	1.71	1.60	375	176	1673	158.91	28.00	73.00
12	0.176270	1.73	1.40	722	60	3374	6.38	37.95	85.00
13	0.200073	1.76	1.59	1639	416	6553	220.46	19.73	76.00
14	0.212524	1.79	1.62	3482	882	12902	483.97	42.97	153.00
15	0.231598	1.81	1.59	7589	1095	25179	523.90	42.69	191.00
16	0.117599	1.74	1.63	7707	2688	57829	1422.26	159.56	457.00
17	0.078995	1.72	1.60	10354	3230	120718	2481.50	371.77	938.00
18	0.078449	1.73	1.37	20565	324	241579	302.81	350.15	1155.00
19	0.061646	1.72	1.64	32320	13471	491968	12128.71	404.87	1233.00
20	0.065800	1.74	1.64	68996	21240	979580	15870.41	285.87	1063.00
21	0.032823	1.69	1.56	68835	11865	2028317	816.87	1050.92	3506.00
22	0.021364	1.67	1.60	89609	32428	4104695	20280.02	1335.34	5030.00
23	0.011244	1.64	1.48	94324	9108	8294284	7929.18	4869.70	10024.00

TABLE 2. $x_0 + x_1x_2$ on the 2-shift: a map which is degree 1, left permutative (hence left closing) and not right closing.

k	Fraction Periodic	kth root of P	kth root of L	P	L	Not-P	Average Period	Average Preperiod	Maximum Preperiod
1	0.500	1.00	1.00	1	1	1	1.00	0.50	1
2	0.750	1.73	1.00	3	1	1	1.00	0.25	1
3	0.500	1.58	1.44	4	3	4	1.75	0.50	1
4	0.687	1.82	1.18	11	2	5	1.75	0.31	1
5	0.812	1.91	1.71	26	15	6	11.00	0.19	1
6	0.468	1.76	1.20	30	3	34	1.66	1.28	3
7	0.500	1.81	1.66	64	35	64	28.34	0.99	4
8	0.667	1.90	1.63	171	52	85	30.39	0.52	3
9	0.306	1.75	1.27	157	9	355	8.89	2.35	8
10	0.261	1.74	1.49	268	55	756	20.18	6.75	18
11	0.387	1.83	1.57	793	143	1255	53.53	3.00	13
12	0.088	1.63	1.16	362	6	3734	1.39	20.61	48
13	0.150	1.72	1.63	1236	611	6956	259.75	20.15	78
14	0.126	1.72	1.51	2068	329	14316	119.61	33.22	132
15	0.091	1.70	1.50	3014	465	29754	414.94	44.45	138
16	0.092	1.72	1.50	6043	728	59493	650.33	101.66	282
17	0.107	1.75	1.68	14145	6783	116927	3918.82	48.16	196
18	0.060	1.71	1.58	15753	4095	246391	3406.78	110.78	396
19	0.072	1.74	1.60	38191	7619	486097	6336.19	142.98	406
20	0.038	1.69	1.54	40396	5780	1008180	1691.96	279.69	780
21	0.018	1.65	1.48	37867	4011	2059285	3961.81	705.45	1777
22	0.017	1.66	1.51	75309	9658	4118995	4527.64	605.57	1770
23	0.017	1.67	1.57	144096	34477	8244512	26857.88	1191.56	2687

TABLE 3. The composition $x_0x_1 + x_2$ followed by $x_0 + x_1x_2$ on the 2-shift: a map which is neither left nor right closing. The polynomial rule for the composition is $x_0x_1 + x_0x_2x_3 + x_1x_2 + x_1x_2x_3 + x_2 + x_3x_4$.

k	Fraction Periodic	kth root of P	kth root of L	P	L	Not-P	Average Period	Average Preperiod	Maximum Preperiod
1	0.5000	1.00	1.00	1	1	1	1.00	0.50	1
2	0.2500	1.00	1.00	1	1	3	1.00	0.75	1
3	0.1250	1.00	1.00	1	1	7	1.00	1.62	2
4	0.3125	1.49	1.41	5	4	11	2.50	0.94	2
5	0.3437	1.61	1.58	11	10	21	9.44	0.66	1
6	0.0156	1.00	1.00	1	1	63	1.00	3.52	5
7	0.2812	1.66	1.54	36	21	92	17.62	1.05	2
8	0.0195	1.22	1.18	5	4	251	3.91	5.65	11
9	0.0546	1.44	1.22	28	6	484	5.82	5.18	11
10	0.1767	1.68	1.46	181	45	843	29.75	2.14	7
11	0.0703	1.57	1.55	144	132	1904	98.08	6.76	19
12	0.0012	1.14	1.12	5	4	4091	1.01	19.25	36
13	0.0556	1.60	1.49	456	182	7736	162.94	18.54	49
14	0.0261	1.54	1.35	428	70	15956	28.54	18.35	55
15	0.0342	1.59	1.45	1121	285	31647	138.58	21.60	58
16	0.0074	1.47	1.47	485	480	65051	430.96	71.09	146
17	0.0160	1.56	1.55	2109	1734	128963	1633.83	51.36	169
18	0.0060	1.50	1.41	1594	549	260550	334.44	70.40	233
19	0.0046	1.50	1.45	2452	1197	521836	834.45	92.00	227
20	0.0058	1.54	1.50	6165	3640	1042411	2700.37	70.21	211
21	0.0017	1.47	1.36	3627	693	2093525	585.86	356.39	817
22	0.0033	1.54	1.46	14004	4147	4180300	3305.59	251.62	864
23	0.0022	1.53	1.53	18746	18538	8369862	18491.96	262.30	900

TABLE 4. The composition $x_0 + x_1$ followed by $x_0 + x_1 x_2$ on the two-shift: algebraic degree 2 followed by degree 1 left permutative.

k	Fraction Periodic	kth root of P	kth root of L	P	L	Not-P	Average Period	Average Preperiod	Maximum Preperiod
1	.5000	1.00	1.00	1	1	1	1.00	0.50	1
2	.2500	1.00	1.00	1	1	3	1.00	1.25	2
3	.5000	1.58	1.44	4	3	4	2.50	0.50	1
4	.3125	1.49	1.41	5	4	11	2.50	1.31	3
5	.3437	1.61	1.58	11	10	21	9.44	0.97	2
6	.4375	1.74	1.61	28	18	36	11.31	0.69	2
7	.0625	1.34	1.32	8	7	120	6.91	4.00	7
8	.0195	1.22	1.18	5	4	251	3.25	6.58	12
9	.1484	1.61	1.58	76	63	436	58.26	3.17	7
10	.0888	1.57	1.52	91	70	933	18.17	7.77	17
11	.0703	1.57	1.46	144	66	1,904	65.35	5.52	14
12	.0576	1.57	1.36	236	42	3,860	24.44	10.98	34
13	.0350	1.54	1.53	287	273	7,905	217.65	11.93	29
14	.0201	1.51	1.39	330	105	16,054	12.65	36.60	74
15	.0123	1.49	1.44	404	255	32,364	179.68	35.36	91
16	.0232	1.58	1.54	1,525	1,008	64,011	272.23	33.28	98
17	.0286	1.62	1.52	3,758	1,377	127,314	913.23	31.04	114
18	.0091	1.54	1.53	2,386	2,250	259,758	2,026.85	55.23	152
19	.0039	1.49	1.47	2,091	1,672	522,197	1,658.11	91.44	251
20	.0015	1.44	1.31	1,635	240	1,046,941	14.16	279.12	575
21	.0046	1.54	1.48	9,650	4,326	2,087,502	461.24	244.11	638
22	.0011	1.47	1.40	4,896	1,848	4,189,408	1,158.45	274.42	647
23	.0027	1.54	1.53	23,461	19,297	8,365,147	18,849.71	269.70	824

TABLE 5. The composition $x_0 + x_1$ followed by the automorphism $U = x_0 + x_{-2}x_1x_2 + x_{-2}x_{-1}x_1x_2$ on the two-shift. U is the involution of the 2-shift which replaces x_0 with $x_0 + 1$ when $x[-2, 2] = 10x_011$.

k	Fraction Periodic	kth root of P	kth root of L	P	L	Not-P	Average Period	Average Preperiod	Maximum Preperiod
1	1.0000	2.00	1.00	2	1	0	1.00	0.00	0
2	.5000	1.41	1.00	2	1	2	1.00	0.50	1
3	.2500	1.25	1.00	2	1	6	1.00	1.12	2
4	.1250	1.18	1.00	2	1	14	1.00	1.62	3
5	.2187	1.47	1.37	7	5	25	1.62	2.03	4
6	.4062	1.72	1.51	26	12	38	4.56	1.20	3
7	.0703	1.36	1.32	9	7	119	6.91	4.65	9
8	.0703	1.43	1.41	18	16	238	1.94	6.98	12
9	.1796	1.65	1.48	92	36	420	15.52	2.55	7
10	.0263	1.39	1.17	27	5	997	1.07	9.08	15
11	.1782	1.70	1.53	365	110	1,683	77.16	3.79	16
12	.0122	1.38	1.30	50	24	4,046	17.51	10.57	26
13	.1049	1.68	1.53	860	260	7,332	199.90	6.20	21
14	.0056	1.38	1.37	93	84	16,291	70.69	22.86	48
15	.0340	1.59	1.43	1,117	225	31,651	117.64	13.52	42
16	.0154	1.54	1.40	1,010	224	64,526	111.24	27.58	68
17	.0135	1.55	1.45	1,770	612	129,302	558.46	41.02	112
18	.0037	1.46	1.33	980	180	261,164	52.93	32.45	107
19	.0078	1.54	1.50	4,125	2,242	520,163	824.24	52.35	168
20	.0011	1.42	1.32	1,227	280	1,047,349	88.00	77.69	196
21	.0008	1.42	1.39	1,731	1,092	2,095,421	29.02	180.81	480
22	.0006	1.43	1.27	2,829	220	4,191,475	85.05	134.13	399
23	.0008	1.46	1.44	6,833	4,462	8,381,775	4,148.57	209.22	699

TABLE 6. The map $x_{-1} + x_0x_1 + x_2$, on the two-shift: linear in both end variables but not algebraic.

k	Fraction Periodic	kth root of P	kth root of L	P	L	Not-P	Average Period	Average Preperiod	Maximum Preperiod
1	.5000	1.00	1.00	1	1	1	1.00	0.50	1
2	.7500	1.73	1.00	3	1	1	1.00	0.25	1
3	.5000	1.58	1.00	4	1	4	1.00	0.88	2
4	.6875	1.82	1.00	11	1	5	1.00	0.31	1
5	.8125	1.91	1.37	26	5	6	2.56	0.19	1
6	.6562	1.86	1.61	42	18	22	7.38	0.86	4
7	.6093	1.86	1.60	78	28	50	17.35	1.16	5
8	.5117	1.83	1.62	131	48	125	27.50	2.08	10
9	.4296	1.82	1.58	220	63	292	43.61	3.39	11
10	.4082	1.82	1.31	418	15	606	5.45	6.91	21
11	.4355	1.85	1.57	892	143	1,156	99.90	12.91	53
12	.3608	1.83	1.36	1,478	42	2,618	11.61	15.59	53
13	.3270	1.83	1.66	2,679	754	5,513	577.86	33.42	123
14	.2167	1.79	1.26	3,552	28	12,832	23.06	79.16	191
15	.2503	1.82	1.48	8,204	385	24,564	303.28	69.75	232
16	.3152	1.86	1.63	20,659	2,528	44,877	1,197.54	48.40	281
17	.1784	1.80	1.55	23,393	1,853	107,679	1,538.93	168.75	464
18	.1821	1.81	1.59	47,760	4,464	214,384	3,208.77	172.00	697
19	.1357	1.80	1.49	71,175	1,957	453,113	1,685.66	352.58	1082
20	.1620	1.82	1.56	169,886	7,976	878,690	5,604.39	258.96	953
21	.1032	1.79	1.52	216,612	7,056	1,880,540	6,344.22	2,389.64	4,363
22	.0902	1.79	1.35	378,612	740	3,815,692	633.16	2,315.42	6,465
23	.0858	1.79	1.58	720,246	39,353	7,668,362	36,059.28	1,760.56	5,984

TABLE 7. The map on the two-shift which is the the automorphism U of Table 5 postcomposed with the left permutative, not right-closing map $x_0 + x_1x_2$.

Map	Tabular rule	Map	Tabular rule
1	0000 1111 0010 1101	17	0011 1001 1100 1100
2	0000 1111 0100 1011	18	0011 1010 0011 1100
3	0001 1100 0011 1110	19	0011 1010 1100 0011
4	0001 1110 0101 1010	20	0011 1100 0101 0011
5	0010 1001 0110 1101	21	0011 1100 0101 1100
6	0010 1101 0000 1111	22	0011 1100 1010 0011
7	0011 0011 0110 0011	23	0011 1100 1010 1100
8	0011 0011 0110 1100	24	0011 1110 0001 1100
9	0011 0011 1001 0011	25	0100 1001 0110 1011
10	0011 0011 1001 1100	26	0100 1011 0000 1111
11	0011 0101 0011 1100	27	0101 1010 0001 1110
12	0011 0101 1100 0011	28	0101 1010 0111 1000
13	0011 0110 0011 0011	29	0110 1011 0100 1001
14	0011 0110 1100 1100	30	0111 1101 0010 1001
15	0011 1000 0111 1100	31	0111 1000 0101 1010
16	0011 1001 0011 0011	32	0111 1100 0011 1000

Table 8. The 32 span 4 onto c.a. of the 2 shift which fix $\dots 000 \dots$ and are not linear in an end variable [8, Table I]. Maps 2, 6, 7 and 16 are one-to-one.

Map	Tabular rule	Map	Tabular rule
1	0001 0111 1110 1000 0001 0111 1111 0000	14	0100 1101 1111 0000 0100 1101 1011 0010
2	0001 1011 0111 0100 1110 0100 1111 0000	15	0110 0001 1010 1011 0110 0001 0110 0111
3	0010 0010 1111 0011 0010 1110 0000 1111	16	0110 1000 0111 1001 0110 0001 1110 1001
4	0010 1001 0110 1101 0100 1001 0110 1011	17	0110 1011 1100 0010 0100 1011 0001 1101
5	0010 1110 0000 1111 0010 1110 1111 0000	18	0111 0001 1011 0010 0111 0001 1000 1110
6	0100 0111 0001 0111 1011 1000 0000 1111	19	0111 0010 1011 0100 0111 0010 0111 1000
7	0100 0111 0100 1011 1000 1011 0100 1011	20	0111 1000 0100 1011 0111 1000 0111 1000
8	0100 1011 1000 0111 0100 1011 0100 1011	21	0111 1000 0100 1011 0111 1000 1011 0100
9	0100 1101 1011 0010 1000 1110 1011 0010	22	0111 1000 0100 1011 0111 1000 1111 0000
10	0100 1101 1011 0010 1100 1100 1011 0010	23	0111 1000 0100 1101 0111 1000 1000 1110
11	0100 1101 1101 0010 0011 0011 1101 0010	24	0111 1011 1000 0100 0100 1011 0000 1111
12	0100 1101 1101 0010 0111 0001 1101 0010	25	0111 1011 1100 0000 0100 1011 0000 1111
13	0100 1101 1101 0010 1111 0000 1101 0010	26	0111 1011 1100 0000 0100 1011 0100 1011

Table 9. 26 irregular span 5 onto maps of the 2 shift which fix $\dots 000 \dots$ and are not linear in an end variable [8, Table I].

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	2.00	2.00	1.00	1.00	2.00	2.00	2.00	1.00	2.00	1.00	1.00	2.00	2.00	1.00	1.00	2.00
2	0.00	1.41	0.00	1.41	1.41	1.41	1.41	1.41	0.00	0.00	0.00	1.41	0.00	1.41	1.41	1.41
3	1.81	1.81	1.81	1.44	0.00	1.81	1.81	1.44	1.81	1.44	1.81	0.00	1.81	1.44	1.44	1.81
4	1.68	1.86	1.68	1.41	1.41	1.86	1.86	1.41	1.68	1.41	1.68	1.41	1.68	0.00	1.68	1.86
5	1.97	1.97	1.82	1.90	1.58	1.97	1.97	1.90	1.97	1.82	1.82	1.58	1.97	1.58	1.90	1.97
6	1.90	1.94	1.81	1.61	1.76	1.94	1.94	1.61	1.90	1.34	1.81	1.76	1.90	1.86	1.81	1.94
7	1.99	1.99	1.83	1.70	1.80	1.99	1.99	1.70	1.99	1.74	1.83	1.80	1.99	1.83	1.70	1.99
8	1.95	1.98	1.81	1.48	1.75	1.98	1.98	1.48	1.95	1.78	1.81	1.75	1.95	1.54	1.88	1.98
9	1.99	1.99	1.86	1.68	1.82	1.99	1.99	1.68	1.99	1.82	1.86	1.82	1.99	1.73	1.86	1.99
10	1.98	1.99	1.76	1.70	1.82	1.99	1.99	1.70	1.98	1.75	1.76	1.82	1.98	1.56	1.65	1.99
11	1.99	1.99	1.70	1.65	1.68	1.99	1.99	1.65	1.99	1.89	1.70	1.68	1.99	1.60	1.90	1.99
12	1.99	1.99	1.51	1.65	1.61	1.99	1.99	1.65	1.99	1.65	1.51	1.61	1.99	1.34	1.75	1.99
13	2.00	2.00	1.70	1.57	1.63	2.00	2.00	1.57	2.00	1.73	1.70	1.63	2.00	1.54	1.68	2.00
14	1.99	1.99	1.74	1.65	1.70	1.99	1.99	1.65	1.99	1.81	1.74	1.70	1.99	1.66	1.74	1.99
15	1.99	1.99	1.71	1.68	1.70	1.99	1.99	1.68	1.99	1.73	1.71	1.70	1.99	1.47	1.77	1.99
16	1.99	1.99	1.74	1.67	1.70	1.99	1.99	1.67	1.99	1.76	1.74	1.70	1.99	1.59	1.67	1.99
17	2.00	2.00	1.67	1.53	1.71	2.00	2.00	1.53	2.00	1.75	1.67	1.71	2.00	1.59	1.61	2.00
18	1.99	1.99	1.71	1.56	1.65	1.99	1.99	1.56	1.99	1.71	1.71	1.65	1.99	1.52	1.63	1.99
19	2.00	2.00	1.73	1.54	1.72	2.00	2.00	1.54	2.00	1.77	1.73	1.72	2.00	1.57	1.69	2.00

TABLE 10. $\nu_k^o(\cdot, S_2)$ for span four onto maps 1-16 of Table 8.

k	$F1$	$F2$	$F3$	$F4$	$F5$	$F6$	$F7$	$F8$	$F9$	$F10$	$F11$	$F12$	$F13$	$F14$	$F15$	$F16$
1	2.00	2.00	1.00	1.00	2.00	2.00	2.00	1.00	2.00	1.00	1.00	2.00	2.00	1.00	1.00	2.00
2	0.00	1.41	0.00	1.41	1.41	1.41	1.41	1.41	0.00	0.00	0.00	1.41	0.00	1.41	1.41	1.41
3	1.81	1.81	1.81	1.44	1.44	1.81	1.81	1.44	1.81	0.00	1.81	1.44	1.81	1.44	0.00	1.81
4	1.41	1.86	1.68	0.00	1.41	1.86	1.86	0.00	1.41	0.00	1.68	1.41	1.41	1.41	1.68	1.86
5	1.97	1.97	1.82	1.58	1.71	1.97	1.97	1.58	1.97	1.71	1.82	1.71	1.97	1.90	1.90	1.97
6	1.86	1.94	1.69	1.86	1.69	1.94	1.94	1.86	1.86	1.69	1.69	1.69	1.86	1.61	0.00	1.94
7	1.99	1.99	1.66	1.83	1.70	1.99	1.99	1.83	1.99	1.60	1.66	1.70	1.99	1.70	1.92	1.99
8	1.93	1.98	1.81	1.54	1.80	1.98	1.98	1.54	1.93	1.41	1.81	1.80	1.93	1.48	1.75	1.98
9	1.99	1.99	1.71	1.73	1.62	1.99	1.99	1.73	1.99	1.77	1.71	1.62	1.99	1.68	1.68	1.99
10	1.95	1.99	1.70	1.56	1.44	1.99	1.99	1.56	1.95	1.79	1.70	1.44	1.95	1.70	0.00	1.99
11	1.99	1.99	1.74	1.60	1.46	1.99	1.99	1.60	1.99	1.59	1.74	1.46	1.99	1.65	1.65	1.99
12	1.97	1.99	1.65	1.34	1.55	1.99	1.99	1.34	1.97	1.63	1.65	1.55	1.97	1.65	1.69	1.99
13	2.00	2.00	1.75	1.54	1.65	2.00	2.00	1.54	2.00	1.53	1.75	1.65	2.00	1.57	1.67	2.00
14	1.98	1.99	1.74	1.66	1.53	1.99	1.99	1.66	1.98	1.72	1.74	1.53	1.98	1.65	1.51	1.99
15	1.99	1.99	1.74	1.47	1.64	1.99	1.99	1.47	1.99	1.68	1.74	1.64	1.99	1.68	1.74	1.99
16	1.99	1.99	1.66	1.59	1.55	1.99	1.99	1.59	1.99	1.66	1.66	1.55	1.99	1.67	1.68	1.99
17	2.00	2.00	1.67	1.59	1.57	2.00	2.00	1.59	2.00	1.74	1.67	1.57	2.00	1.53	1.69	2.00
18	1.99	1.99	1.63	1.52	1.61	1.99	1.99	1.52	1.99	1.70	1.63	1.61	1.99	1.56	1.61	1.99
19	2.00	2.00	1.69	1.57	1.51	2.00	2.00	1.57	2.00	1.54	1.69	1.51	2.00	1.54	1.63	2.00

TABLE 11. $\nu_k^o(\cdot, S_2)$ for span 4 maps 1-16 of Table 8, postcomposed with the flip map F (given by rule $x_0 + 1$).

k	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
1	1.00	1.00	2.00	2.00	1.00	2.00	1.00	1.00	2.00	2.00	1.00	1.00	2.00	2.00	1.00	1.00
2	0.00	1.41	0.00	1.41	1.41	0.00	0.00	1.41	1.41	0.00	0.00	1.41	0.00	0.00	0.00	0.00
3	0.00	0.00	0.00	0.00	1.44	0.00	1.81	0.00	0.00	1.81	1.44	1.44	0.00	0.00	0.00	1.81
4	0.00	1.68	1.68	1.41	1.68	1.68	1.68	1.68	1.41	1.68	1.41	0.00	1.68	1.68	0.00	1.68
5	1.71	1.90	1.71	1.58	1.90	1.71	1.82	1.90	1.58	1.97	1.82	1.58	1.71	1.58	1.71	1.82
6	1.69	0.00	1.34	1.76	1.81	1.34	1.69	0.00	1.76	1.90	1.34	1.86	1.34	1.51	1.69	1.69
7	1.60	1.92	1.54	1.80	1.70	1.54	1.66	1.92	1.80	1.99	1.74	1.83	1.54	1.60	1.60	1.66
8	1.41	1.75	1.41	1.75	1.88	1.41	1.81	1.75	1.75	1.95	1.78	1.54	1.41	1.58	1.41	1.81
9	1.77	1.68	1.74	1.82	1.86	1.74	1.71	1.68	1.82	1.99	1.82	1.73	1.74	1.44	1.77	1.71
10	1.79	0.00	1.72	1.82	1.65	1.72	1.70	0.00	1.82	1.98	1.75	1.56	1.72	1.54	1.79	1.70
11	1.59	1.65	1.41	1.68	1.90	1.41	1.74	1.65	1.68	1.99	1.89	1.60	1.41	1.54	1.59	1.74
12	1.63	1.69	1.59	1.61	1.75	1.59	1.65	1.69	1.61	1.99	1.65	1.34	1.59	1.54	1.63	1.65
13	1.53	1.67	1.66	1.63	1.68	1.66	1.75	1.67	1.63	2.00	1.73	1.54	1.66	1.60	1.53	1.75
14	1.72	1.51	1.44	1.70	1.74	1.44	1.74	1.51	1.70	1.99	1.81	1.66	1.44	1.51	1.72	1.74
15	1.68	1.74	1.58	1.70	1.77	1.58	1.74	1.74	1.70	1.99	1.73	1.47	1.58	1.50	1.68	1.74
16	1.66	1.68	1.64	1.70	1.67	1.64	1.66	1.68	1.70	1.99	1.76	1.59	1.64	1.49	1.66	1.66
17	1.74	1.69	1.59	1.71	1.61	1.59	1.67	1.69	1.71	2.00	1.75	1.59	1.59	1.40	1.74	1.67
18	1.70	1.61	1.46	1.65	1.63	1.46	1.63	1.61	1.65	1.99	1.71	1.52	1.46	1.48	1.70	1.63
19	1.54	1.63	1.60	1.72	1.69	1.60	1.69	1.63	1.72	2.00	1.77	1.57	1.60	1.42	1.54	1.69

TABLE 12. $\nu_k^o(\cdot, S_2)$ for span 4 onto maps 17-32 of Table 8.

k	$F17$	$F18$	$F19$	$F20$	$F21$	$F22$	$F23$	$F24$	$F25$	$F26$	$F27$	$F28$	$F29$	$F30$	$F31$	$F32$
1	1.00	1.00	2.00	2.00	1.00	2.00	1.00	1.00	2.00	2.00	1.00	1.00	2.00	2.00	1.00	1.00
2	0.00	1.41	0.00	1.41	1.41	0.00	0.00	1.41	1.41	0.00	0.00	1.41	0.00	0.00	0.00	0.00
3	1.44	1.44	1.44	1.44	0.00	1.44	1.81	1.44	1.44	1.81	0.00	1.44	1.44	1.44	1.44	1.81
4	1.41	1.68	1.68	1.41	1.68	1.68	1.68	1.68	1.41	1.41	0.00	1.41	1.68	1.68	1.41	1.68
5	1.82	1.90	1.58	1.71	1.90	1.58	1.82	1.90	1.71	1.97	1.71	1.90	1.58	1.71	1.82	1.82
6	1.34	1.81	1.61	1.69	0.00	1.61	1.81	1.81	1.69	1.86	1.69	1.61	1.61	1.76	1.34	1.81
7	1.74	1.70	1.70	1.70	1.92	1.70	1.83	1.70	1.70	1.99	1.60	1.70	1.70	1.74	1.74	1.83
8	1.78	1.88	1.65	1.80	1.75	1.65	1.81	1.88	1.80	1.93	1.41	1.48	1.65	1.62	1.78	1.81
9	1.82	1.86	1.60	1.62	1.68	1.60	1.86	1.86	1.62	1.99	1.77	1.68	1.60	1.72	1.82	1.86
10	1.75	1.65	1.61	1.44	0.00	1.61	1.76	1.65	1.44	1.95	1.79	1.70	1.61	1.52	1.75	1.76
11	1.89	1.90	1.69	1.46	1.65	1.69	1.70	1.90	1.46	1.99	1.59	1.65	1.69	1.61	1.89	1.70
12	1.65	1.75	1.49	1.55	1.69	1.49	1.51	1.75	1.55	1.97	1.63	1.65	1.49	1.62	1.65	1.51
13	1.73	1.68	1.66	1.65	1.67	1.66	1.70	1.68	1.65	2.00	1.53	1.57	1.66	1.58	1.73	1.70
14	1.81	1.74	1.59	1.53	1.51	1.59	1.74	1.74	1.53	1.98	1.72	1.65	1.59	1.53	1.81	1.74
15	1.73	1.77	1.53	1.64	1.74	1.53	1.71	1.77	1.64	1.99	1.68	1.68	1.53	1.56	1.73	1.71
16	1.76	1.67	1.69	1.55	1.68	1.69	1.74	1.67	1.55	1.99	1.66	1.67	1.69	1.56	1.76	1.74
17	1.75	1.61	1.63	1.57	1.69	1.63	1.67	1.61	1.57	2.00	1.74	1.53	1.63	1.58	1.75	1.67
18	1.71	1.63	1.55	1.61	1.61	1.55	1.71	1.63	1.61	1.99	1.70	1.56	1.55	1.58	1.71	1.71
19	1.77	1.69	1.65	1.51	1.63	1.65	1.73	1.69	1.51	2.00	1.54	1.54	1.65	1.57	1.77	1.73

TABLE 13. $\nu_k^o(\cdot, S_2)$ for span 4 maps 17-32 of Table 8, postcomposed with the flip map F (given by rule $x_0 + 1$).

k	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1.00	1.00	2.00	2.00	1.00	2.00	2.00	2.00	1.00	1.00	1.00	1.00	1.00
2	0.00	0.00	0.00	1.41	1.41	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
3	1.81	1.44	1.81	1.81	0.00	1.44	1.44	1.81	1.81	1.44	1.44	0.00	0.00
4	1.41	0.00	0.00	1.86	0.00	1.41	1.68	0.00	0.00	0.00	1.68	1.68	1.41
5	1.90	1.82	1.90	1.82	1.71	1.97	1.71	1.97	1.90	1.37	1.71	1.71	1.82
6	1.61	0.00	1.51	1.69	0.00	1.69	1.76	1.86	1.81	1.61	1.51	1.61	0.00
7	1.77	1.80	1.80	1.77	1.74	1.88	1.90	1.99	1.45	1.85	1.90	1.70	1.92
8	1.62	1.70	1.75	1.70	1.48	1.72	1.81	1.90	1.80	0.00	1.78	1.72	1.70
9	1.89	1.76	1.74	1.90	1.90	1.91	1.91	1.99	1.79	1.58	1.76	1.77	1.80
10	1.80	1.58	1.25	1.83	1.78	1.81	1.91	1.95	1.67	0.00	1.76	1.68	1.90
11	1.66	1.69	1.75	1.73	1.78	1.85	1.92	1.99	1.67	1.66	1.80	1.51	1.48
12	1.71	1.75	1.78	1.84	1.68	1.85	1.84	1.97	1.77	1.70	1.68	1.73	1.69
13	1.73	1.72	1.79	1.73	1.72	1.87	1.93	2.00	1.72	1.69	1.80	1.75	1.84
14	1.66	1.61	1.73	1.73	1.63	1.81	1.91	1.98	1.57	1.54	1.69	1.68	1.74
15	1.66	1.71	1.60	1.73	1.74	1.85	1.92	1.99	1.78	1.67	1.70	1.68	1.76
16	1.68	1.64	1.74	1.71	1.72	1.79	1.93	1.98	1.54	1.65	1.67	1.49	1.75
17	1.69	1.53	1.73	1.68	1.72	1.84	1.91	2.00	1.73	1.59	1.73	1.65	1.69
18	1.68	1.46	1.69	1.68	1.71	1.83	1.91	1.99	1.65	1.59	1.57	1.65	1.67
19	1.67	1.61	1.68	1.67	1.69	1.81	1.93	2.00	1.71	1.63	1.66	1.71	1.73

TABLE 14. $\nu_k^o(\cdot, S_2)$ for span five onto maps 1-13 of Table 9.

k	14	15	16	17	18	19	20	21	22	23	24	25	26
1	1.00	2.00	2.00	2.00	1.00	1.00	1.00	1.00	1.00	1.00	2.00	2.00	2.00
2	0.00	1.41	1.41	0.00	1.41	1.41	0.00	0.00	0.00	0.00	0.00	0.00	0.00
3	1.81	1.44	0.00	1.44	1.44	1.44	0.00	0.00	0.00	0.00	1.81	1.81	1.81
4	0.00	1.68	1.86	0.00	1.41	1.68	0.00	1.41	0.00	1.68	0.00	0.00	0.00
5	1.82	1.82	1.82	1.71	1.71	1.90	1.71	1.58	1.71	1.58	1.97	1.97	1.97
6	1.81	1.86	1.76	1.61	1.51	1.51	0.00	1.69	1.81	1.81	1.69	1.69	1.69
7	1.80	1.85	1.92	1.60	1.70	1.83	1.83	1.54	1.70	1.54	1.99	1.99	1.99
8	1.41	1.83	1.70	1.72	1.83	1.86	1.68	1.68	1.54	1.68	1.81	1.81	1.86
9	1.84	1.84	1.76	1.76	1.71	1.78	1.78	1.74	1.84	1.87	1.99	1.99	1.99
10	1.86	1.85	1.83	1.65	1.76	1.82	1.79	1.52	1.67	1.72	1.90	1.90	1.93
11	1.57	1.78	1.81	1.88	1.81	1.75	1.77	1.64	1.79	1.54	1.99	1.99	1.99
12	1.57	1.83	1.84	1.66	1.64	1.80	1.62	1.73	1.73	1.67	1.94	1.94	1.94
13	1.71	1.71	1.68	1.73	1.72	1.85	1.61	1.78	1.79	1.65	2.00	2.00	2.00
14	1.76	1.73	1.72	1.67	1.66	1.80	1.57	1.73	1.75	1.60	1.95	1.95	1.96
15	1.71	1.76	1.79	1.72	1.77	1.78	1.72	1.65	1.66	1.52	1.99	1.99	1.99
16	1.71	1.71	1.71	1.73	1.69	1.78	1.71	1.51	1.65	1.55	1.97	1.97	1.97
17	1.74	1.66	1.63	1.64	1.71	1.78	1.64	1.65	1.67	1.68	2.00	2.00	2.00
18	1.71	1.68	1.68	1.67	1.62	1.73	1.65	1.61	1.52	1.62	1.98	1.98	1.98
19	1.68	1.69	1.70	1.70	1.63	1.75	1.70	1.66	1.65	1.60	2.00	2.00	2.00

TABLE 15. $\nu_k^o(\cdot, S_2)$ for span five onto maps 14-26 of Table 9.

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1.00	1.00	2.00	2.00	1.00	1.00	1.00	2.00	1.00	2.00	2.00	1.00	1.00	2.00	2.00	1.00
2	1.41	0.00	1.41	0.00	0.00	0.00	0.00	0.00	1.41	1.41	1.41	0.00	1.41	0.00	0.00	0.00
3	1.44	0.00	1.44	0.00	0.00	1.44	1.44	1.81	0.00	0.00	1.44	1.81	0.00	1.81	1.44	1.44
4	0.00	1.41	1.41	1.68	1.68	0.00	1.86	1.41	0.00	0.00	1.41	1.41	1.41	1.41	1.68	1.86
5	1.37	1.37	1.58	1.71	1.82	1.90	1.37	1.71	1.90	1.71	1.58	1.82	1.37	1.71	1.37	1.90
6	1.51	1.51	1.51	1.34	1.51	1.34	1.34	0.00	0.00	1.51	0.00	1.61	1.34	0.00	1.34	0.00
7	1.74	1.88	1.54	1.60	1.66	1.60	1.77	1.54	1.70	1.66	0.00	1.74	1.80	1.54	1.60	1.32
8	1.48	1.58	1.65	1.62	1.41	1.65	0.00	1.62	1.62	1.58	1.68	1.54	0.00	1.62	1.68	0.00
9	1.62	1.78	1.74	1.58	1.74	1.58	1.68	0.00	1.48	1.48	1.44	1.73	1.44	0.00	1.64	1.60
10	1.34	1.62	1.40	1.54	1.66	1.58	1.52	1.68	1.61	1.62	1.34	1.47	1.34	1.68	1.34	1.50
11	1.58	1.69	1.43	1.64	1.51	1.76	1.67	1.60	1.55	1.53	1.60	1.66	1.58	1.60	1.57	1.67
12	1.54	1.44	1.60	1.51	1.63	1.74	1.46	1.56	0.00	1.54	1.54	1.44	1.52	1.56	1.38	1.38
13	1.61	1.66	1.62	1.65	1.56	1.77	1.59	1.63	1.32	1.66	1.61	1.47	1.57	1.63	1.68	1.60
14	1.64	1.62	1.52	1.57	1.53	1.80	1.55	1.40	1.55	1.41	1.60	1.55	1.48	1.40	1.55	1.65
15	1.60	1.74	1.60	1.38	1.62	1.73	1.58	1.62	1.52	1.52	1.50	1.49	1.66	1.62	1.66	1.61
16	1.48	1.73	1.47	1.53	1.50	1.60	1.51	1.59	1.49	1.44	1.29	1.58	1.57	1.59	1.49	1.57
17	1.46	1.69	1.58	1.59	1.62	1.64	1.61	1.47	1.60	1.58	1.56	1.55	1.46	1.47	1.63	1.36
18	1.56	1.72	1.55	1.49	1.55	1.52	1.46	1.50	1.51	1.52	1.54	1.45	1.56	1.50	1.52	1.45
19	1.56	1.66	1.56	1.55	1.52	1.64	1.45	1.60	1.47	1.51	1.55	1.56	1.52	1.60	1.59	1.48

TABLE 16. ν_k^o for 16 left permutative span 5 maps on the two-shift.

k	Fraction Periodic	kth root of P	kth root of L	P	L	Not-P	Average Period	Average Preperiod	Maximum Preperiod
1	.5000	1.00	1.00	1	1	1	1.00	0.50	1
2	.2500	1.00	1.00	1	1	3	1.00	1.25	2
3	.1250	1.00	1.00	1	1	7	1.00	0.88	1
4	.0625	1.00	1.00	1	1	15	1.00	2.06	3
5	.3437	1.61	1.37	11	5	21	4.12	1.28	3
6	.4843	1.77	1.51	31	12	33	4.47	0.55	2
7	.1718	1.55	1.54	22	21	106	20.69	1.54	4
8	.1289	1.54	1.48	33	24	223	16.81	2.66	6
9	.0546	1.44	1.22	28	6	484	1.62	5.23	11
10	.1572	1.66	1.63	161	140	863	117.99	2.28	8
11	.0380	1.48	1.32	78	22	1,970	4.54	8.70	22
12	.0280	1.48	1.30	115	24	3,981	11.91	7.74	20
13	.0318	1.53	1.49	261	182	7,931	174.97	13.62	31
14	.0133	1.46	1.38	218	98	16,166	49.23	11.07	31
15	.0159	1.51	1.49	521	420	32,247	151.23	14.69	45
16	.0044	1.42	1.38	289	176	65,247	26.10	44.74	83
17	.0088	1.51	1.47	1,157	782	129,915	606.20	22.33	91
18	.0073	1.52	1.49	1,930	1368	260,214	852.90	28.42	102
19	.0049	1.51	1.47	2,604	1710	521,684	773.84	31.12	91
20	.0005	1.37	1.28	561	140	1,048,015	33.23	91.21	256
21	.0014	1.46	1.40	3,109	1344	2,094,043	734.01	59.95	192
22	.0009	1.45	1.41	4,038	2112	4,190,266	1,791.24	97.89	279
23	.0008	1.47	1.43	7,315	3910	8,381,293	3,648.59	128.94	492

TABLE 17. This is a composition of two span 4 onto maps of the two-shift from Table 8, map 8 followed by map 30.

k	Fraction Periodic	kth root of P	kth root of L	P	L	Not-P	Average Period	Average Preperiod	Maximum Preperiod
1	1.00	3.00	2.00	3	2	0	1.67	0.00	0.00
2	1.00	3.00	2.44	9	6	0	4.56	0.00	0.00
3	1.00	3.00	2.46	27	15	0	9.30	0.00	0.00
4	0.11	1.73	1.56	9	6	72	4.56	0.89	1.00
5	1.00	3.00	2.77	243	165	0	121.13	0.00	0.00
6	1.00	3.00	2.80	729	486	0	334.54	0.00	0.00
7	1.00	3.00	2.57	2187	742	0	401.62	0.00	0.00
8	0.01	1.73	1.58	81	40	6480	33.50	4.24	9.00
9	1.00	3.00	2.24	19683	1469	0	1185.85	0.00	0.00
10	1.00	3.00	2.76	59049	25865	0	22737.63	0.00	0.00
11	1.00	3.00	2.92	177147	131857	0	109208.21	0.00	0.00
12	0.00	1.76	1.67	909	486	530532	239.27	45.51	133.00
13	1.00	3.00	2.79	1594323	631605	0	291222.95	0.00	0.00

TABLE 18. This map on the 3-shift is an automorphism W followed by the degree 9 linear map $x_0 + x_2$, where $W = x_0 + 2x_0x_1x_1 + 2x_0x_1 + x_1x_1 + x_1$. Let π denote the permutation on $\{0, 1, 2\}$ which transposes 0 and 2. Then $(Wx)_0 = x_0$ if $x_1 \neq 1$ and $(Wx)_0 = \pi(x_0)$ if $x_1 = 1$.

k	Fraction Periodic	kth root of P	kth root of L	P	L	Not-P	Average Period	Average Preperiod	Maximum Preperiod
1	.3333	1.00	1.00	1	1	2	1.00	1.00	2
2	.5555	2.23	1.00	5	1	4	1.00	0.56	2
3	.2592	1.91	1.00	7	1	20	1.00	1.78	3
4	.2098	2.03	1.00	17	1	64	1.00	3.17	8
5	.4362	2.54	2.09	106	40	137	27.65	1.08	4
6	.2208	2.33	1.51	161	12	568	6.79	2.71	9
7	.0932	2.13	1.66	204	35	1,983	5.89	13.89	38
8	.0391	2.00	1.00	257	1	6,304	1.00	27.02	67
9	.1667	2.45	1.60	3,283	72	16,400	48.89	13.41	52
10	.0299	2.11	1.62	1,770	130	57,279	62.67	55.38	163
11	.0224	2.12	1.89	3,972	1122	173,175	593.34	99.23	297
12	.0164	2.13	1.40	8,729	60	522,712	12.56	88.45	222
13	.0076	2.06	1.81	12,117	2366	1,582,206	2,228.50	676.85	1,504

TABLE 19. The map $x_0 + x_1x_2$ on the 3-shift: still degree 1, left permutative, not right closing.

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