Evaluating a Framework for Learning Non-routine Problem-Solving in Mathematics

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Abstract: An important goal of mathematics education is supporting students' ability to tackle novel, non-routine problems. However, typical mathematics instruction in U.S. classrooms often places emphasis on rote learning, without exposing students to the process of deriving a formula or alternative solutions. We propose a framework, *Try-See-Tell-Do*, which combines several evidence-based learning strategies to help students develop non-routine problem-solving skills. Following the framework, a student would first *try* a non-routine problem on their own, *see* a worked example solution to the problem, self-explain (*tell*) the solution they saw, and finally practice with two novel non-routine problems (*do*). Through a series of eight think-aloud interviews, where high school students applied the framework to solve combinatorics problems, we found that students could recognize the instructional value of the learning activities. Additionally, we noted students' tendency to rely on memorized formulas and derived actionable insights on improving the framework to reduce this behavior.

Introduction

The ability to solve non-routine problems plays an increasingly important role in modern workplaces (Neubert et al., 2015), where well-defined routines are insufficient. However, in U.S. classrooms, non-routine problems are often viewed as targeting only high-performing students, and even with these students, there is a lack of empirical evidence for which instructional strategy, or combination of strategies, is most effective (Gavaz et al., 2021). Towards addressing this gap, we propose and evaluate an instructional framework, *Try-See-Tell-Do*, intended to develop non-routine problem solving skills by combining several evidence-based learning science techniques: solving a problem on one's own (*try*), seeing a worked example of this same problem (*see*), self-explaining a solution to the problem (*tell*), and finally practicing with related problems (*do*). These activities are based on well-established benefits of attempting to invent a solution to a new problem before receiving instruction on it (Schwartz & Martin, 2004), worked examples as a means of scaffolding (Paas et al., 2003), and self-explanation as a trigger of generative processing that supports deep learning (Chi et al., 1989).

While each component of the proposed framework has strong conceptual and empirical support, these learning science principles have rarely been applied to non-routine problems or in tandem. Thus, our goal in this research is to evaluate the effectiveness of the *Try-See-Tell-Do* framework as a whole. To this end, we formulated the conceptual strategies that students were expected to learn and designed learning activities based on these strategies and the framework pipeline. Then, we conducted think-aloud interview studies to record students' performance and feedback as they tried out the framework. In what follows, we describe each phase of *Try-See-Tell-Do* in greater detail and formulate the primary take-aways from our qualitative analyses.

Conceptualizing how students learn non-routine problem solving

In preparation for developing the *Try-See-Tell-Do* framework, we solved 70 non-routine mathematics problems from the textbook by Johnson & Herr (2011) and consolidated the techniques introduced in the textbook into eight strategies. Four strategies are for "preprocessing," used to create a mental representation of the problem: *Use Smaller Number*, *Specify*, *List Requirements* and *Identify Target Quantities*. Four strategies are "solving" strategies, used to come up with a solution: *List Number Sequence, Solve Smaller Subproblem, Solve Complementary Problem, Create Equations*. For example, the non-routine problem of finding the last digit in the number 2^{57} can be solved with the preprocessing strategy *Use Smaller Number,* followed by the solving strategy *List Number Sequence*. Specifically, one could list the final digits of smaller exponents of 2 – 2^1 ends in 2, 2^2 ends in 4, 2^3 ends in 8, 2^4 ends in 6, 2^5 ends in 2 – to identify the recurring pattern 2, 4, 8, 6. Such a pattern in turn implies that 2^{57} should end in 2, without needing to directly compute this number.

In a practical setting, students would have the opportunity to experiment with different combinations of preprocessing steps and solving strategies to construct a solution. In the scope of this work, however, we wanted to first evaluate whether the framework pipeline is effective. Therefore, we only had students work with one preprocessing step, *Use Smaller Number*, and one solving strategy, *List Number Sequence*, in our study. We selected three problems from Johnson and Herr (2011)'s textbook that have distinct contexts and involve different number sequences, but can all be solved with this combination of preprocessing and solving strategies:

- **Q1**: You and your friends are organizing a tournament, where each person will play one match against each other. There are five players in total - A, B, C, D, E. How many matches will be played in total?
- **Q2**: Alexis, Bart, Chuck, Jerry, Timmy, Kim, and Dariah are all called in to a radio show to get free tickets to a concert. How many possible orders in which their calls could have been received are there?
- **Q3**: In a movie theater, there are ten rows of seats. The first row contains one seat, the second three seats, the third five seats, the fourth seven seats, and so on. How many seats are there in total?

One of the three problems (Q1) was assigned to the *Try, See,* and *Tell* stages, where students could make an attempt at solving (*Try*) before seeing the worked example (*See*) and explaining a solution (*Tell*). The remaining problems (Q2, Q3) played the role of near- and far-transfer practice problems in the *Do* stage, after students had learned from Q1's solution. We selected high school students as our interview participants since they may have had some, but probably limited, experience with combinatorics, which is typically covered in the first year of undergraduate study in the U.S. In this way, the participants were not expected to have prior formal exposure to the problems and would consider them as non-routine.

Methods

We recruited eight high school students (6 males, 2 females, aged 15-18) in the U.S. to participate in our online think-aloud interviews. Participation was completely voluntary and participants were compensated $25 after the interview. First, participants were asked about their math background, including their grade level, classes taken, and general experience with math. Then they were instructed on how the think-aloud process would be conducted, along with a think-aloud demo on a simple math problem, carried out by the interviewer. The interviews each proceeded based on the four stages of our instructional framework. In the *Try* stage, participants were asked to solve a non-routine problem (Q1) on their own. They then went to the *See* stage and saw a worked example solution to the same problem, Q1. To promote interactive learning, we presented each step as a multiple-choice question on a web interface (1). After answering each question, participants would see a feedback message about their correctness. Participants concluded the *See* stage by reviewing the entirety of the worked example solution, before moving to the *Tell* stage. Here, they were asked to treat the interviewer as a novice and tell them how to solve Q1, using either the solution they derived in the *Try* stage or the one they saw in the *See* stage. Next, the *Do* stage presented two more problems (Q2 and Q3) for participants to solve on their own to measure near and far transfer. Finally, we asked participants about their thoughts on the overall framework.

With permission from the participants and their parents, we recorded the interview audio along with the participants' draft work in each stage. Then we transcribed and performed a narrative inquiry among three researchers on the audio data, focusing on participants' self-reported math background, their feedback on each stage, and their evaluation of the overall framework. We also investigated whether the first three stages provided sufficient learning support for participants to tackle the near-transfer problem (Q2) and far-transfer problem (Q3) in the final *Do* stage, as well as any notable difficulties that participants encountered throughout the interview. From this point, we denote each participant with an anonymous ID, ranging from P1 to P8.

Analysis and Results

There were five rising sophomores and three rising juniors who took part in the study. Five participants had taken classes up to Algebra 2 and three up to Advanced Placement Calculus. All participants indicated that they enjoyed learning math because they liked to challenge themselves or because they had a positive learning experience.

Try stage

In this stage, participants solved the problem Q1 on their own. Five participants were familiar with the underlying combinatorial formula of the problem and were quick to point out the correct answer, "5 choose 2," which equals to 10. The remaining three did not mention any connection to combinatorics, likely because they were not exposed to this field. Among them, one arrived at the correct answer by listing out all the 10 matches. The other two attempted to visualize the problem by drawing the tournament layout in a graph; however, these participants incorrectly considered "A vs B" and "B vs A" as different matches, leading to an overestimation. When asked about the rationale for their chosen approach, participants either mentioned that they had seen a similar problem or that they felt the visualization and enumeration approach was the most natural.

See stage

In this stage, participants reviewed a worked example of the *List Number Sequence* strategy for Q1, accompanied by seven multiple-choice questions. The first question asked about whether it's better to consider all the participating teams at once, or start with a smaller number of teams. The expected answer is the latter, which reflects the *Use Smaller Number* preprocessing strategy, but seven out of eight participants selected the former, which aligns with their approach in the *Try* stage. P6 noted that both options could lead to the correct answer, so the question text should be clearer about why one option is preferred over another. After this point, all participants picked up on the spirit of the solution and were able to correctly answer subsequent questions. When asked about their preference, five participants preferred their original approach (applying the combinatorial formula), on the grounds that the worked example took longer to carry out than their original solution and may not work for more complicated problems. At the same time, they acknowledged that this bottom-up approach could be helpful to beginner students who have no knowledge of combinatorics – as P4 stated, "*It shouldn't be set as the be-all-end-all thinking process as people obviously process thoughts differently, but it's a great baseline to start off the process if people have no idea how to think of problems like this*." The remaining three participants were more welcoming of the worked example, noting that seeing the explanations helped them better understand the solution; P2 further compared this process to their online learning experience: "*Khan Academy shows the steps but doesn't force you to work through it, so this is more helpful*."

Tell stage

In this stage, participants were asked to teach a novice about solving Q1, using either their original solution in the *Try* stage or the worked example in the *See* stage. Five participants taught with their original solution because they were more comfortable with it (P1, P3) or because they believed it is more effective and generalizable (P6, P7, P8). The remaining three chose to explain the worked example, rather than their original combinatorial approach. When asked about their rationale for switching, P5 indicated that the worked example presented a new approach to them, so they'd like to teach it to understand it better. Likewise, P4 considered seeing the fine-grained worked example a new experience -- "*it shows you a process of how to teach others to think, it's a very streamlined process and very natural, but I wouldn't have thought of it before.*" When reflecting on their experience, all participants indicated that this *Tell* stage is the most helpful in the framework. P5 was very familiar with the combinatorial formula solution; thus, having to explain the solution to a novice forced them to think about why the formula works and delve more into the underlying principles. P2 also made a reference to the inductive process "*starting small, building big in a systematic way,*" which they found particularly useful.

Do stage

The first problem (Q2) in this stage was solved successfully by four out of eight participants (P5, P6, P7, P8). These participants were able to immediately point out the answer, which is 7! = 5040, as the problem context is naturally how permutation is defined. Notably, one participant, P2, realized that the problem is related to a combinatorial formula but could not recall the formula -- "*I just can't remember what I actually did with this problem because we had something like this before as well.*" The remaining participants either proposed an incorrect answer (7 x 7 = 49) after some trial-and-error or gave up without a solution. The second *Do* problem (Q3) observed more success, where six participants (P2, P4, P5, P6, P7, P8) arrived at the correct answer but did not start from smaller cases, instead considering the whole problem setting. However, participant P5, after mentioning the general formula, also attempted a second solution that followed the *List Number Sequence* strategy from the *See* stage; P5 was the only participant that did so. The remaining two participants failed because they either made a calculation mistake or could not recall an appropriate formula.

When asked about whether the previous stages (*Try*, *See*, *Tell*) helped with their performance on the *Do* stage, P2 noted that these stages helped orient their thoughts and have a clear direction of what to do. The remaining participants indicated that the previous stages had little effect on them, either because they had already seen all three problems given in the interview (P5, P6, P7, P8), or because they didn't apply the strategy in the *See* stage to solve subsequent problems (P1, P3, P4). However, some participants felt that they could have learned from the previous stages if given more time to digest the problem and solution content: "*now that I'm thinking about it, it helped me just now realize that I could minimize the number*" (P1).

Discussion and Conclusion

This work explores a general framework for training high school students in solving non-routine problems. While the framework did not yield clear learning benefits as measured by performance on the practice problems, it reveals important insights into students' problem-solving behaviors and suggests potential future avenues.

Our first observation is that all participants had a strong tendency to appeal to formulas during their problem solving. When these attempts failed, they would then resort to solving the problem by brute force (i.e., list all possible configurations) and would give up if this approach was not fruitful. This behavior occurred most frequently in Q2, where there are too many configurations to manually enumerate. Consistent with our findings, Stigler et al. (2010) has reported that, when facing a novel problem that requires flexibility, U.S. students tend to apply memorized procedures or formulas incorrectly rather than develop new solutions.

To combat this issue, we devised the *Try-See-Tell-Do* framework to highlight a general strategy for discovering a solution to a problem, without relying on rote memorization. We hypothesized that the *See* stage, with its detailed worked example, would clarify the conceptual workflow for students, and that subsequent stages would reinforce this workflow through self-explanation, near-transfer and far-transfer practice. Our interview results showed that participants did find the worked example useful, especially for less experienced students, which is consistent with research showing worked examples are most beneficial for novices (Paas et al., 2003). However, most participants were not able to capture the underlying heuristic and apply them to new problems. On the other hand, the *Tell* stage was positively received by all participants, who indicated that it was a novel and helpful experience. Although students thinking an approach is helpful and actually *showing* it is helpful through actual use are two different outcomes, this finding does at least point to past research that shows self-explanation following a worked example is a powerful and robust mechanism to support learning (Chi et al., 1989). For our next experiment, we plan to improve the *See* stage with an interactive visualization of the worked example, which can be effective at maintaining engagement (Liang & Sedig, 2010), while also highlight the core take-aways. As observed from past studies on problem-solving (Arslan & Yazgan, 2015; Nguyen et al., 2020), the visualization approach aligns well with how students tend to explore the solution space when attempting novel problems

Finally, we remark that our interviews were conducted on a small sample of students who had either solid or excellent backgrounds in mathematics. To promote broader applicability, additional evaluation of the *Try-See-Tell-Do* framework is needed, especially with lower-performing students, for whom the multi-layered instructional support may be more beneficial. Nevertheless, we have uncovered students' reliance on formula application as a central issue in problem solving and identified room for improvement in the implementation of our learning strategies. We envision that, through iterative refinement, the framework can be deployed as a digital learning platform with support for personalized feedback to promote non-routine problem-solving skills at scale.

Endnotes

(1) A demo of the worked example prototype is available at: https://bit.ly/3cb25ov.

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