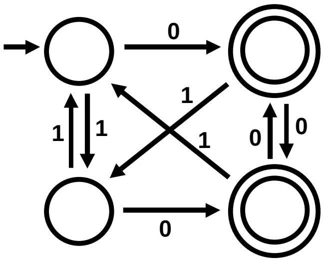
MINIMIZING DFAs

THURSDAY SEP 8

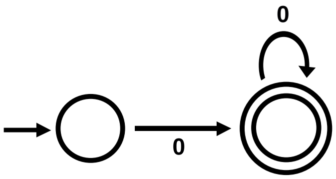
IS THIS **MINIMAL?**



THEOREM

For every regular language L, there exists a unique (up to re-labeling of the states) minimal DFA M such that L = L(M)

NOT TRUE FOR NFAs



EXTENDING δ

Given DFA $M = (Q, \Sigma, \delta, q_0, F)$ extend δ to $\hat{\delta} : Q \times \Sigma^* \to Q$ as follows:

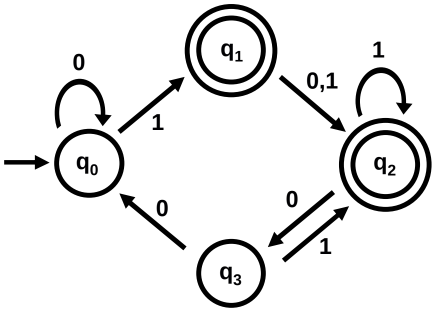
$$\hat{\delta}(q, \varepsilon) = q$$

$$\hat{\delta}(q, \sigma) = \delta(q, \sigma)$$

$$\hat{\delta}(q, w_1 \ldots w_{k+1}) = \delta(\hat{\delta}(q, w_1 \ldots w_k), w_{k+1})$$

A string $w \in \Sigma^*$ distinguishes states q_1 from q_2 if

$$\hat{\delta}(q_1, w) \in F \iff \hat{\delta}(q_2, w) \notin F$$



Fix $M = (Q, \Sigma, \delta, q_0, F)$ and let $p, q, r \in Q$

Definition:

$p \sim q$ iff p is indistinguishable from q

$p \not\sim q$ iff p is distinguishable from q

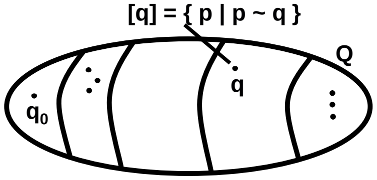
Proposition: ~ is an equivalence relation

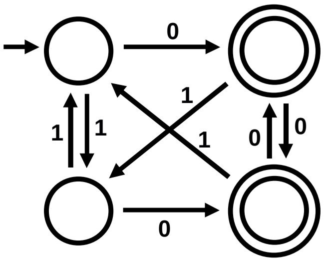
- **$p \sim p$ (reflexive)**
- **$p \sim q \Rightarrow q \sim p$ (symmetric)**
- **$p \sim q$ and $q \sim r \Rightarrow p \sim r$ (transitive)**

Fix $M = (Q, \Sigma, \delta, q_0, F)$ and let $p, q, r \in Q$

Proposition: ~ is an equivalence relation so ~ partitions the set of states of M into disjoint equivalence classes

$[q] = \{ p \mid p \sim q \}$





Algorithm MINIMIZE

Input: DFA M

Output: DFA M_{MIN} such that:

$M \equiv M_{MIN}$

M_{MIN} has no inaccessible states

M_{MIN} is irreducible

||

states of M_{MIN} are pairwise distinguishable

Theorem: M_{MIN} is the unique minimum

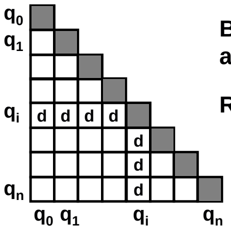
Idea: States of M_{MIN} will be blocks of equivalent states of M

TABLE-FILLING ALGORITHM

Input: DFA $M = (Q, \Sigma, \delta, q_0, F)$

Output: (1) $D_M = \{ (p,q) \mid p,q \in Q \text{ and } p \not\equiv q \}$

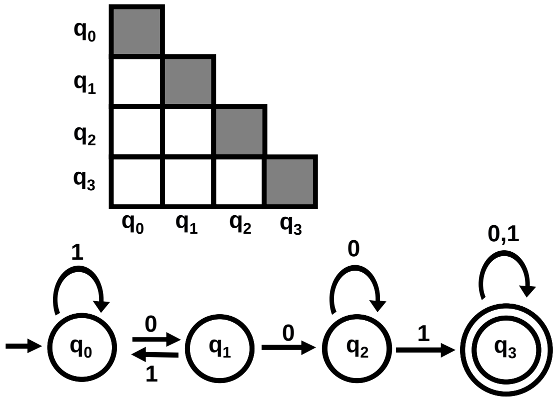
(2) $E_M = \{ [q] \mid q \in Q \}$



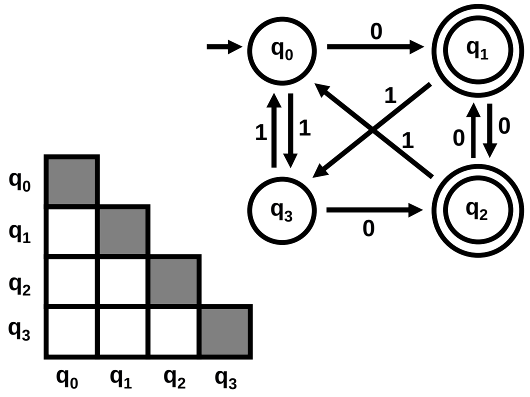
Base Case: p accepts and q rejects $\Rightarrow p \not\equiv q$

Recursion:

$$\begin{array}{l} p \xrightarrow{\sigma} p' \\ \qquad\quad \not\equiv \;\Rightarrow\; p \not\equiv q \\ q \xrightarrow{\sigma} q' \end{array}$$



	q_0	q_1	q_2	q_3
q_0	▨			
q_1		▨		
q_2			▨	
q_3				▨



	q_0	q_1	q_2	q_3
q_0	▨			
q_1		▨		
q_2			▨	
q_3				▨

If 2 states are not distinguished by table-filling algorithm, then they are equivalent

Proof (by contradiction):

Suppose there exist states p, q not distinguished by the T-F algorithm, but $p \not\sim q$

Then there exists w such that:

$$\hat{\delta}(p, w) \in F \text{ and } \hat{\delta}(q, w) \notin F$$

$$w = \sigma w', \text{ where } \sigma \in \Sigma$$

Let $p' = \delta(p,\sigma)$ and $q' = \delta(q,\sigma)$

Then (p', q') is also a bad pair

Algorithm MINIMIZE

Input: DFA M

Output: DFA M_{MIN}

(1) Remove all inaccessible states from M

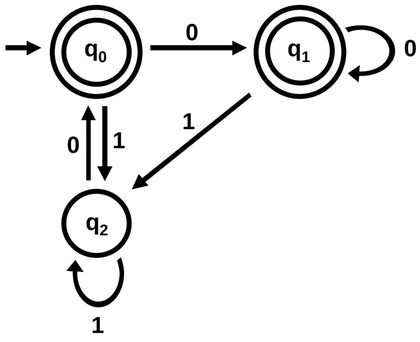
(2) Apply Table-Filling algorithm to get $E_M = \{ [q] \mid q \text{ is an accessible state of } M \}$

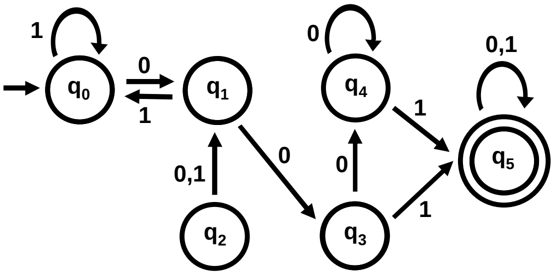
$$M_{MIN} = (Q_{MIN}, \Sigma, \delta_{MIN}, q_{0\,MIN}, F_{MIN})$$

$$Q_{MIN} = E_M,\ \ q_{0\,MIN} = [q_0],\ \ F_{MIN} = \{ [q] \mid q \in F \}$$

$$\delta_{MIN}([q],\sigma) = [\ \delta(q,\sigma)\ ]$$

MINIMIZE





M_{MIN} is the unique minimal DFA equivalent to M

Suppose $M' \equiv M_{MIN}$, and M′ has no inaccessible states and is irreducible

Claim: There exists a 1-1 correspondence between M′ and M_{MIN}

Proof: We will construct a map recursively

Base Case: $q_{0\ MIN} \to q_0'$

Recursive Step: If

$$\begin{tikzcd} p \arrow[r] \arrow[d, "\sigma"] & p' \arrow[d, "\sigma"] \\ q & q' \end{tikzcd}$$

Then $q \to q'$

Base Case: $q_{0\ MIN} \to q_0'$

Recursive Step: If

$$\begin{tikzcd} p \arrow[r] \arrow[d, "\sigma"] & p' \arrow[d, "\sigma"] \\ q & q' \end{tikzcd}$$

Then $q \to q'$

We need to prove:

- The map is defined everywhere
- The map is well defined
- The map is a bijection

Base Case: $q_{0\,MIN} \to q_0'$

Recursive Step: If $$\begin{tikzcd} p \arrow[r] \arrow[d, "\sigma"] & p' \arrow[d, "\sigma"] \\ q & q' \end{tikzcd}$$ Then $q \to q'$

The map is defined everywhere

That is, for all $q \in M_{MIN}$ there is a $q' \in M'$ such that $q \to q'$

If $q \in M_{MIN}$, there is w such that
$\hat{\delta}_{MIN}(q_{0\,MIN},w) = q$

Let $q' = \hat{\delta}'(q_0',w)$

Base Case: $q_{0\,MIN} \to q_0'$

Recursive Step: If $$\begin{tikzcd} p \arrow[r] \arrow[d, "\sigma"] & p' \arrow[d, "\sigma"] \\ q & q' \end{tikzcd}$$ Then $q \to q'$

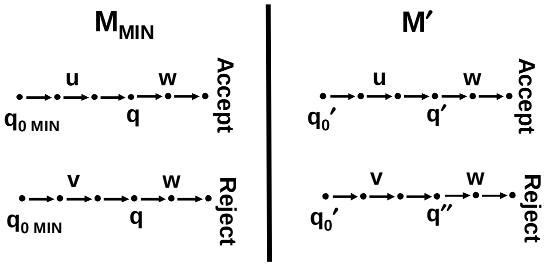
The map is well defined

Suppose there exist q' and q'' such that $q \to q'$ and $q \to q''$

We show that q' and q'' are indistinguishable, so it must be that $q' = q''$

Suppose there exist q' and q'' such that $q \to q'$ and $q \to q''$

Suppose q' and q'' are not indistinguishable



Base Case: $q_{0\,MIN} \to q_0'$

Recursive Step: If $$\begin{tikzcd} p \arrow[r] \arrow[d, "\sigma"] & p' \arrow[d, "\sigma"] \\ q & q' \end{tikzcd}$$ Then $q \to q'$

The map is onto

Base Case: $q_{0\,MIN} \to q_0'$

Recursive Step: If

$$\begin{tikzcd} p \arrow[r] \arrow[d, "\sigma"] & p' \arrow[d, "\sigma"] \\ q & q' \end{tikzcd}$$

Then $q \to q'$

The map is 1-1

How do we prove two regular expressions are equivalent?

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Finish Chapter 1 of the book for next time