PSPACE-COMPLETENESS

THURSDAY NOVEMBER 10

$$\text{PSPACE} = \bigcup_{k \in N} \text{SPACE}(n^k)$$

Definition: Language B is PSPACE-complete if:

1. $B \in$ PSPACE
2. Every A in PSPACE is poly-time reducible to B (i.e. B is PSPACE-hard)

Definition: A fully quantified Boolean formula (or a sentence) is a Boolean formula in which every variable is quantified

$$\exists x \exists y\, [\, x \vee \neg y \,]$$

$$\forall x\, [\, x \vee \neg x \,]$$

$$\forall x\, [\, x \,]$$

$$\forall x \exists y\, [\, (x \vee y) \wedge (\neg x \vee \neg y) \,]$$

TQBF = { ϕ | ϕ is a true fully quantified Boolean formula}

Theorem: TQBF is PSPACE-complete

Claim: Every language A in PSPACE is polynomial time reducible to TQBF

We build a poly-time reduction from A to TQBF

The reduction turns a string w into a fully quantified Boolean formula ϕ that simulates the PSPACE machine for A on w

Let M be a deterministic TM that decides A in space n^k How do we know M exists?

A tableau for M on w is an table whose rows are the configurations of the computation of M on input w

#	q_0	w_1	w_2	…	w_n	□	…	□	#
#									#
#									#

We now design ϕ so that a satisfying assignment to the variables corresponds to M accepting w

Given two collections of variables denoted c and d representing two configurations and a number t > 0, we construct a formula $\phi_{c,d,t}$

If we assign c and d to actual configurations, $\phi_{c,d,t}$ will be true if and only if M can go from c to d in t steps

We let $\phi = \phi_{c_{start}, c_{accept}, h}$, where $h = 2^{df(n)}$ for a constant d chosen so that M has less than $2^{df(n)}$ possible configurations on an input of length n

If t = 1, we can easily construct $\phi_{c,d,t}$:

$\phi_{c,d,t}$ = "c equals d" or "d follows from c in a single step of M"

"c equals d" can be expressed by writing a Boolean formula saying that each of the variables representing c is equal to the corresponding one in d

"d follows from c in a single step of M" can be expressed using 2 x 3 windows as in the Cook-Levin theorem

If t > 1, we construct $\phi_{c,d,t}$ recursively:

$$\phi_{c,d,t} = \exists m\, [\phi_{c,m,t/2} \wedge \phi_{m,d,t/2}]$$

$\exists x_1 \exists x_2 \ldots \exists x_l$

How long is this formula?

Since O(t) is too long, we modify the formula to be:

$$\phi_{c,d,t} = \exists m \forall a,b[\, [(a,b)=(c,m) \vee (a,b)=(m,d)] \rightarrow [\, \phi_{a,b,t/2}\,]\,]$$

TQBF = { ϕ | ϕ is a true fully quantified Boolean formula}

Theorem: TQBF is PSPACE-complete

PSPACE is frequently called the class of games because many popular games are PSPACE-Complete

THE FORMULA GAME

…is played between two players, E and A

Given a fully quantified Boolean formula

$$\exists y \forall x \, [\, (x \vee y) \wedge (\neg x \vee \neg y) \,]$$

E chooses values for variables quantified by ∃

A chooses values for variables quantified by ∀

Start at the leftmost quantifier

E wins if the resulting formula is true

A wins otherwise

$$\forall x \exists y \, [\, (x \vee y) \wedge (\neg x \vee \neg y) \,]$$

$$\exists x \exists y \, [\, x \vee \neg y \,]$$

FG = { ϕ | Player E has a winning strategy in ϕ }

Theorem: FG is PSPACE-Complete

Proof:

FG = TQBF

GEOGRAPHY

Two players take turns naming cities from anywhere in the world

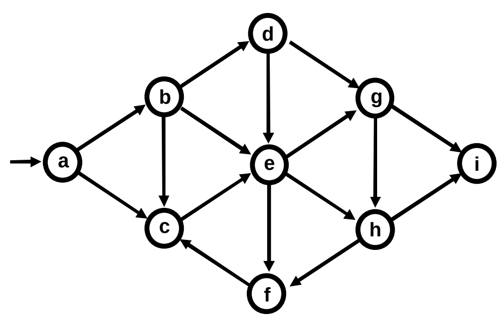
Each city chosen must begin with the same letter that the previous city ended with

Cities cannot be repeated

Austin → Nashua → Albany → York

Whoever cannot name any more cities looses

GENERALIZED GEOGRAPHY



GG = { (G,b) | Player I has a winning strategy for the generalized geography game played on graph G starting at node b }

Theorem: GG is PSPACE-Complete

GG $\in$ PSPACE

M(G,b):

1. Remove node b and all arrows touching it to get to a new graph G_1

2. For each of the nodes $b_1, b_2, \ldots, b_k$ that b originally pointed at, recursively call $M(G_1, b_i)$

3. If any of these reject, Player I has a winning strategy, so accept. Otherwise, reject

GG IS PSPACE-HARD

We show that FG $\leq_P$ GG

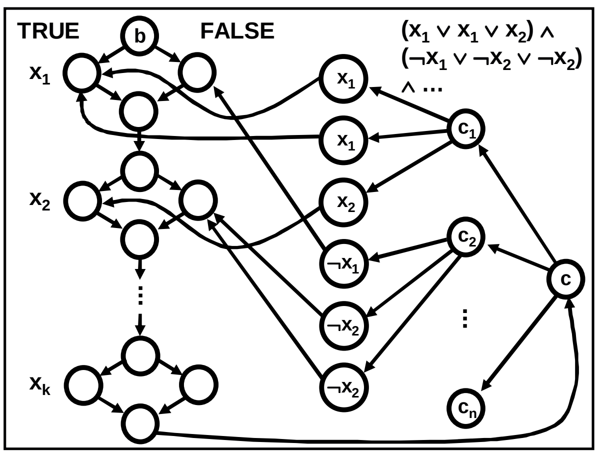
We convert a formula ϕ into (G,b) such that:

Player E has winning strategy in ϕ if and only if Player I has winning strategy in (G,b)

For simplicity we assume ϕ is of the form:

$$\phi = \exists x_1 \forall x_2 \exists x_3 \ldots \exists x_k [\psi]$$

where ψ is in cnf



Question: Is Chess PSPACE complete?

n x n GO, chess and checkers can be shown to be PSPACE-hard

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Read Chapter 8.3 of the book for next time