NP-COMPLETENESS: THE COOK-LEVIN THEOREM

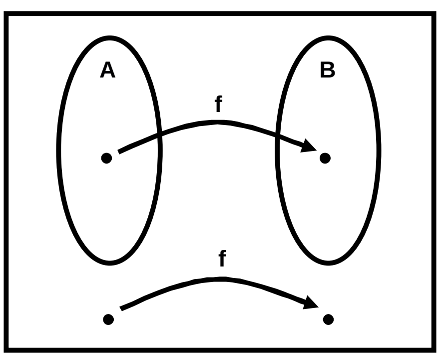
TUESDAY OCTOBER 25

POLY-TIME REDUCIBILITY

A language A is polynomial time reducible to language B, written $A \leq_P B$, if there is a polynomial time computable function $f : \Sigma^* \to \Sigma^*$, where for every w,

$$w \in A \Leftrightarrow f(w) \in B$$

f is called a polynomial time reduction of A to B



$$w \in A \Leftrightarrow f(w) \in B$$

Definition: A language B is NP-complete if:

1. $B \in NP$
2. Every A in NP is poly-time reducible to B (i.e. B is NP-hard)

Theorem (Cook-Levin): SAT is NP-complete

Proof:

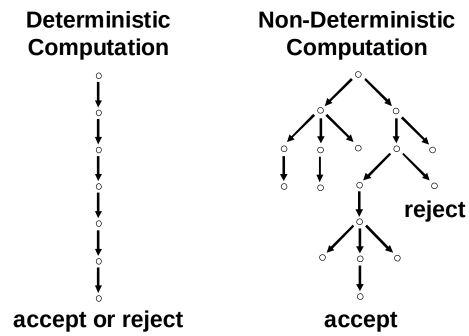
(1) SAT $\in$ NP

(2) Every language A in NP is polynomial time Reducible to SAT

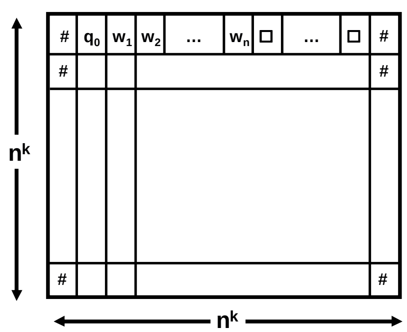
We build a poly-time reduction from A to SAT

The reduction turns a string w into a Boolean formula ϕ that simulates the NP machine for A on w

Let N be a non-deterministic TM that decides A in time n^k



A tableau for N on w is an $n^k \times n^k$ table whose rows are the configurations of a branch of the computation of N on input w



A tableau is accepting if any row of the tableau is an accepting configuration

Determining whether N accepts w is equivalent to determining whether an accepting tableau for N on w exists

VARIABLES

Let $C = Q \cup \Gamma \cup \{ \# \}$

Each of the $(n^k)^2$ entries of a tableau is a cell

The cell in row i and column j is called cell[i,j]

For each i and j ($1 \leq i, j \leq n^k$) and for each $s \in C$ we have a variable $x_{i,j,s}$

These are the variables of ϕ and represent the contents of the cells

If $x_{i,j,s}$ takes on the value 1, it means that cell[i,j] contains s

$$x_{i,j,s} = 1$$

means

$$\text{cell}[\, i, j \,] = s$$

We now design ϕ so that a satisfying assignment to the variables corresponds to an accepting tableau for N on w

The formula ϕ will be the AND of four parts:

$\phi = \phi_{cell} \wedge \phi_{start} \wedge \phi_{accept} \wedge \phi_{move}$

- ϕ_{cell} ensures that for each i,j exactly one $x_{i,j,s}$ is true
- ϕ_{start} ensures that the first row of the table is the starting configuration of N on w
- ϕ_{accept} ensures that an accepting configuration occurs in the table
- ϕ_{move} ensures that every row is a configuration that legally follows from the previous

$$\phi_{cell} = \bigwedge_{1 \leq i, j \leq n^k} \left[\left(\bigvee_{s \in C} x_{i,j,s} \right) \wedge \left(\bigwedge_{\substack{s,t \in C \\ s \neq t}} (\neg x_{i,j,s} \vee \neg x_{i,j,t}) \right) \right]$$

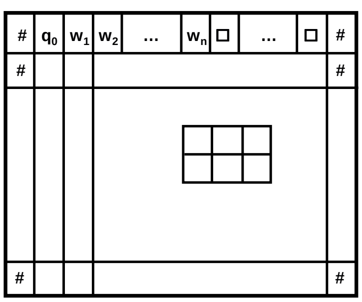
at least one variable is turned on (first part); at most one variable is turned on (second part)

$$\phi_{start} = x_{1,1,\#} \wedge x_{1,2,q_0} \wedge x_{1,3,w_1} \wedge x_{1,4,w_2} \wedge \cdots \wedge x_{1,n+2,w_n} \wedge x_{1,n+3,\square} \wedge \cdots \wedge x_{1,n^k-1,\square} \wedge x_{1,n^k,\#}$$

$$\phi_{accept} = \bigvee_{1 \le i,\, j \le n^k} x_{i,j,q_{accept}}$$

ϕ_{move} ensures that every row is a configuration that legally follows from the previous

It does so by ensuring that each 2×3 window of cells is legal



If $\delta(q_1,a) = \{(q_1,b,R)\}$ and $\delta(q_1,b) = \{(q_2,c,L), (q_2,a,R)\}$ which of the following windows are legal:

a	q_1	b
q_2	a	c

a	q_1	b
q_1	a	a

a	a	q_1
a	a	b

#	b	a
#	b	a

a	b	a
a	b	q_2

b	q_1	b
q_2	b	q_2

a	b	a
a	a	a

a	q_1	b
a	a	q_2

b	b	b
c	b	b

Claim: If the top row of the table is the start configuration and every window is legal, then each row of the table is a configuration that legally follows the preceding one.

$$\phi_{move} = \bigwedge_{1 \le i,\, j \le n^k} (\text{ the } (i, j) \text{ window is legal })$$

the (i, j) window is legal =

$$\bigvee_{\substack{a_1, \ldots, a_6 \\ \text{is a legal window}}} (x_{i-1,j,a_1} \wedge x_{i,j,a_2} \wedge x_{i+1,j,a_3} \wedge x_{i-1,j+1,a_4} \wedge x_{i,j+1,a_5} \wedge x_{i+1,j+1,a_6})$$

WHAT'S THE **LENGTH OF ϕ?**

$$\phi = \phi_{cell} \wedge \phi_{start} \wedge \phi_{accept} \wedge \phi_{move}$$

$$\phi_{cell} = \bigwedge_{1 \le i,\, j \le n^k} \left[\left(\bigvee_{s \in C} x_{i,j,s} \right) \wedge \left(\bigwedge_{\substack{s,t \in C \\ s \neq t}} (\neg x_{i,j,s} \vee \neg x_{i,j,t}) \right) \right]$$

$$\phi_{start} = x_{1,1,\#} \wedge x_{1,2,q_0} \wedge x_{1,3,w_1} \wedge x_{1,4,w_2} \wedge \cdots \wedge x_{1,n+2,w_n} \wedge x_{1,n+3,\square} \wedge \cdots \wedge x_{1,n^k-1,\square} \wedge x_{1,n^k,\#}$$

$$\phi_{accept} = \bigvee_{1 \le i,\, j \le n^k} x_{i,j,q_{accept}}$$

$$\phi_{move} = \bigwedge_{1 \le i,\, j \le n^k} (\text{ the } (i, j) \text{ window is legal })$$

the (i, j) window is legal =

$$\bigvee_{\substack{a_1, \ldots, a_6 \\ \text{is a legal window}}} (x_{i-1,j,a_1} \wedge x_{i,j,a_2} \wedge x_{i+1,j,a_3} \wedge x_{i-1,j+1,a_4} \wedge x_{i,j+1,a_5} \wedge x_{i+1,j+1,a_6})$$

What changes if w changes but n stays the same?

$$\phi = \phi_{cell} \wedge \phi_{start} \wedge \phi_{accept} \wedge \phi_{move}$$

Theorem (Cook-Levin): SAT is NP-complete

Corollary: SAT $\in$ P if and only if P = NP

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Read Chapters 7.4 and 7.5 of the book for next time