

THE ARITHMETIC HIERARCHY

TUESDAY OCTOBER 11

COMPUTABLE FUNCTION

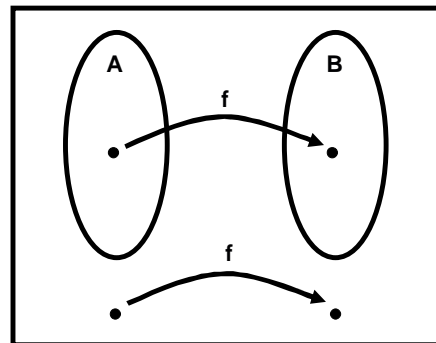
$f : \Sigma^* \rightarrow \Sigma^*$ is a computable function if some Turing machine M , on every input w , halts with just $f(w)$ on its tape

MAPPING REDUCIBILITY

A language A is mapping reducible to language B , written $A \leq_m B$, if there is a computable function $f : \Sigma^* \rightarrow \Sigma^*$, where for every w ,

$$w \in A \Leftrightarrow f(w) \in B$$

f is called a reduction from A to B



Theorem: If $A \leq_m B$ and B is decidable, then A is decidable

Proof: Let M decide B and let f be a reduction from A to B

We build a machine N that decides A as follows:

On input w :

1. Compute $f(w)$
2. Run M on $f(w)$

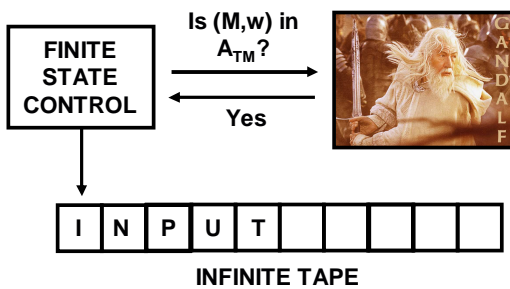
A_{TM}

$HALT_{TM}$

$FIN = \{ M \mid M \text{ is a TM and } L(M) \text{ is finite} \}$

Are all of these equally hard?

ORACLE TMs



ORACLE MACHINES

An oracle is a set B to which the TM may pose membership questions and always receive correct answers after a finite amount of time

This makes sense even if B is not decidable!

We say A is semi-decidable in B if there is an oracle TM M with oracle B that semi-decides A

We say A is decidable in B if there is an oracle TM M with oracle B that decides A

HALT_{TM} is decidable in A_{TM}

On input (M,w) , decide if M halts on w as follows:

1. Ask the oracle for A_{TM} whether M accepts w . If yes, then ACCEPT
2. Switch the accept and reject states of M to get M' . Ask the oracle for A_{TM} whether M' accepts w . If yes, then ACCEPT
3. REJECT

A_{TM} is decidable in HALT_{TM}

On input (M,w) , decide if M accepts w as follows:

Ask the oracle for HALT_{TM} whether M halts on w . If yes, then output $M(w)$. If no, then REJECT.

A Turing Reduces to B

We say A is decidable in B if there is an oracle TM M with oracle B that decides A

$$A \leq_T B$$

\leq_T is transitive

\leq_T **VERSUS** \leq_m

Theorem: If $A \leq_m B$ then $A \leq_T B$

Proof:

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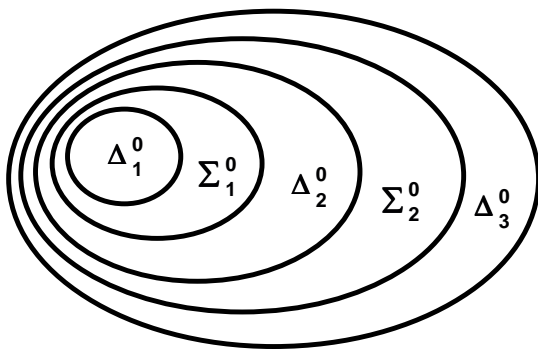
$\Delta_1^0 = \{ \text{decidable sets} \}$

$\Sigma_1^0 = \{ \text{semi-decidable sets} \}$

$\Sigma_{n+1}^0 = \{ \text{sets semi-decidable in some } B \in \Sigma_n^0 \}$

$\Delta_{n+1}^0 = \{ \text{sets decidable in some } B \in \Sigma_n^0 \}$

$\Pi_n^0 = \{ \text{complements of sets in } \Sigma_n^0 \}$



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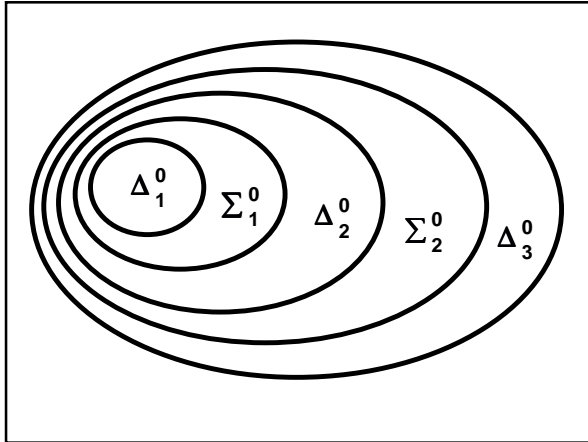
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Theorem: A language A is semi-decidable if and only if there exists a decidable binary predicate $R(x,y)$ such that:

$$A = \{ x \mid \exists y R(x,y) \}$$

Proof:

(1) If $A = \{ x \mid \exists y R(x,y) \}$ then A is semi-decidable

(2) If A is semi-decidable, then $A = \{ x \mid \exists y R(x,y) \}$

$\Sigma_1^0 =$ languages of the form $\{ x \mid \exists y R(x,y) \}$

$\Pi_1^0 =$ languages of the form $\{ x \mid \forall y R(x,y) \}$

$$\Delta_1^0 = \Sigma_1^0 \cap \Pi_1^0$$

$\Sigma_2^0 =$ languages $\{ x \mid \exists y_1 \forall y_2 R(x,y_1,y_2) \}$

$\Pi_2^0 =$ languages $\{ x \mid \forall y_1 \exists y_2 R(x,y_1,y_2) \}$

$$\Delta_2^0 = \Sigma_2^0 \cap \Pi_2^0$$

$\Sigma_n^0 = \text{languages } \{ x \mid \exists y_1 \forall y_2 \exists y_3 \dots Q y_n R(x, y_1, \dots, y_n) \}$

$\Pi_n^0 = \text{languages } \{ x \mid \forall y_1 \exists y_2 \forall y_3 \dots Q y_n R(x, y_1, \dots, y_n) \}$

$\Delta_n^0 = \Sigma_n^0 \cap \Pi_n^0$

$\Sigma_1^0 = \text{languages of the form } \{ x \mid \exists y R(x, y) \}$

Show that A_{TM} is in Σ_1^0

$\Pi_2^0 = \text{languages of the form } \{ x \mid \forall y \exists z R(x, y, z) \}$

Show that $TOTAL = \{ M \mid M \text{ halts on every input} \}$
is in Π_2^0

$\Pi_1^0 = \text{languages of the form } \{ x \mid \forall y R(x, y) \}$

Show that $EMPTY = \{ M \mid L(M) = \emptyset \}$ is in Π_1^0

Σ_2^0 = languages of the form $\{ x \mid \exists y \forall z R(x,y,z) \}$

Show that $\text{FIN} = \{ M \mid L(M) \text{ is finite} \}$ is in Σ_2^0

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Read handout for next time