15-453 Homework # 9

1. CNF (20 Points)

Show that TQBF restricted to formulas where the part following the quantifiers is in conjunctive normal form is still PSPACE-complete.

2. G0-MOKU (20 Points)

The Japanese game *go-moku* is played by two players, "X" and "O," on a 19×19 grid. Players take turns placing markers, and the first player to achieve 5 of his or her markers consecutively in a row, column, or diagonal, is the winner. Consider this game generalized to an $n \times n$ board. Let $GM = \{P \mid P$ is a position in generalized go-moku, where player "X" has a winning strategy $\}$. By *position* we mean a board with markers placed on it, such as may occur in the middle of a play of the game. Show that $GM \in$ PSPACE.

3. More Space is Good (40 Points)

Definition: $f(n) = o(g(n))$ means that, for any real number $c > 0$, a number n_0 exists, where $f(n) < cg(n)$ for all $n \geq n_0$.

Definition: A function $f : \mathcal{N} \rightarrow \mathcal{N}$, where $f(n) \geq \log n$ is called *good* if there exists a Turing machine that maps 1^n to the binary representation of $f(n)$ in space $O(f(n))$. (Most of the functions you have seen in your life, such as $f(n) = \log(n)$, $f(n) = n^2$, etc., are good.)

Problem: *Prove that for any good function $f : \mathcal{N} \rightarrow \mathcal{N}$, there exists a language A that is decidable in space $O(f(n))$ but not in space $o(f(n))$.*

Hint: Consider the following machine B:

$B =$"On input w:

1. Let n be the length of w.
2. Compute $f(n)$ and mark off this much tape. If later stages ever attempt to use more tape than this, *reject*.
3. If w is not of the form $M10^*$ for some TM M, *reject*.
4. If we reach this point, w must be of the form $M10^*$. Simulate M on w while counting the number of steps used in the simulation. If the counter ever exceeds $2^{f(n)}$, *reject*.
5. If M accepts, *reject*. If M rejects, *accept*."

Let A be the language that B decides. Show that B runs in space $O(f(n))$, and show that no machine using space $o(f(n))$ can decide A. (Note that if M exists such that M decides A in space $o(f(n))$, then there is an n_0 such that B's simulation of M will run to completion as long as the input has length n_0 or more. Why? Consider running B on $M10^{n_0}$.)