15-453 Homework # 6

1. A_{TM}
(20 Points)

Prove that all semi-decidable languages mapping reduce to A_{TM}.

2. PCP
(30 Points)

Let $\text{AMBIG}_{CFG} = \{G \mid G \text{ is an ambiguous CFG }\}$. Show that AMBIG_{CFG} is undecidable by reducing from PCP. That is, given an instance

$$P = \left\{\frac{t_1}{b_1}, \frac{t_2}{b_2}, \ldots, \frac{t_k}{b_k}\right\}$$

of the PCP, construct a CFG G with the rules:

$$S \to T|B$$
$$T \to t_1Ta_1|\cdots|t_kTa_k|\epsilon$$
$$B \to b_1Ba_1|\cdots|b_kBa_k|\epsilon$$

where $a_1, \ldots, a_k$ are new terminal symbols. Prove that this reduction works.

3. The Arithmetic Hierarchy
(30 Points)

Prove the following three statements:

- A set A is in Σ_n^0 if and only if there exists a decidable $(n+1)$-ary predicate R such that:

  $$A = \{x|\exists y_1 \forall y_2 \exists y_3 \ldots Qy_n\ R(x, y_1, \ldots, y_n)\}$$

  where $Q = \exists$ if n is odd and $\forall$ if n is even.
- A set A is in Π_n^0 if and only if there exists a decidable $(n+1)$-ary predicate R such that:

  $$A = \{x|\forall y_1 \exists y_2 \forall y_3 \ldots Qy_n\ R(x, y_1, \ldots, y_n)\}$$

  where $Q = \forall$ if n is odd and $\exists$ if n is even.
- $\Delta_n^0 = \Sigma_n^0 \cap \Pi_n^0$.