Machine Learning for Signal Processing

Fundamentals of Linear Algebra

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Instructor: Bhiksha Raj
Overview

• Vectors and matrices
• Basic vector/matrix operations
• Various matrix types
• Projections
• Fundamentals of Linear Algebra, Gilbert Strang

• Important to be very comfortable with linear algebra
  – Appears repeatedly in the form of Eigen analysis, SVD, Factor analysis
  – Appears through various properties of matrices that are used in machine learning
    – Often used in the processing of data of various kinds
    – Will use sound and images as examples

• Today’s lecture: Definitions
  – Very small subset of all that’s used
  – Important subset, intended to help you recollect
Incentive to use linear algebra

• Simplified notation!
  \[ \mathbf{x}^T \cdot \mathbf{A} \cdot \mathbf{y} \quad \longleftrightarrow \quad \sum_j y_j \sum_i x_i a_{ij} \]

• Easier intuition
  – Really convenient geometric interpretations

• Easy code translation!

```plaintext
for i=1:n
  for j=1:m
    c(i)=c(i)+y(j)*x(i)*a(i,j)
  end
end
```

\[ \mathbf{C} = \mathbf{x} \cdot \mathbf{A} \cdot \mathbf{y} \]
And other things you can do

- Manipulate Data
- Extract information from data
- Represent data..
- Etc.

Rotation + Projection + Scaling + Perspective

From Bach’s Fugue in Gm

Decomposition (NMF)
 Scalars, vectors, matrices, ...

• A *scalar* $a$ is a number
  - $a = 2$, $a = 3.14$, $a = -1000$, etc.

• A *vector* $a$ is a linear arrangement of a collection of scalars

$$a = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \quad a = \begin{bmatrix} 3.14 \\ -32 \end{bmatrix}$$

• A *matrix* $A$ is a rectangular arrangement of a collection of scalars

$$A = \begin{bmatrix} 3.12 & -10 \\ 10.0 & 2 \end{bmatrix}$$
Vectors in the abstract

• Ordered collection of numbers
  – Examples: [3 4 5], [a b c d], ..
  – [3 4 5] != [4 3 5] \(\rightarrow\) Order is important

• Typically viewed as identifying (the path from origin to) a location in an N-dimensional space
Vectors in reality

- Vectors usually hold sets of numerical attributes
  - X, Y, Z coordinates
    - [1, 2, 0]
  - [height(cm) weight(kg)]
    - [175 72]
  - A location in Manhattan
    - [3av 33st]
  - A series of daily temperatures
  - Samples in an audio signal
  - Etc.
Matrices

• Matrices can be square or rectangular

\[
S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad R = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}, \quad M = \begin{bmatrix} \vdots \end{bmatrix}
\]

– Can hold data
  • Images, collections of sounds, etc.
  • Or represent *operations* as we shall see

– A matrix can be vertical stacking of row vectors

\[
R = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}
\]

– Or a horizontal arrangement of column vectors

\[
R = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}
\]
Dimensions of a matrix

- The matrix size is specified by the number of rows and columns

\[
c = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \quad r = \begin{bmatrix} a & b & c \end{bmatrix}
\]

- \( c = 3 \times 1 \) matrix: 3 rows and 1 column
- \( r = 1 \times 3 \) matrix: 1 row and 3 columns

\[
S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad R = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}
\]

- \( S = 2 \times 2 \) matrix
- \( R = 2 \times 3 \) matrix
- Pacman = 321 \times 399 \) matrix
Representing an image as a matrix

- 3 pacmen
- A 321 x 399 matrix
  - Row and Column = position
- A 3 x 128079 matrix
  - Triples of x,y and value
- A 1 x 128079 vector
  - “Unraveling” the matrix

- Note: All of these can be recast as the matrix that forms the image
  - Representations 2 and 4 are equivalent
    - The position is not represented
Basic arithmetic operations

• Addition and subtraction
  – Element-wise operations

\[
a + b = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{bmatrix} \quad a - b = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} - \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_1 - b_1 \\ a_2 - b_2 \\ a_3 - b_3 \end{bmatrix}
\]

\[
A + B = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}
\]
Vector Operations

- Operations tell us how to get from origin to the result of the vector operations
  - \((3,4,5) + (3,-2,-3) = (6,2,2)\)
Operations example

Adding random values to different representations of the image
Vector norm

• Measure of how long a vector is:
  – Represented as $|x|$
  
  \[
  \left\| \begin{bmatrix} a & b & \ldots \end{bmatrix} \right\| = \sqrt{a^2 + b^2 + \ldots}
  \]

• Geometrically the shortest distance to travel from the origin to the destination
  – As the crow flies
  – Assuming Euclidean Geometry

• MATLAB syntax: \text{norm}(x)
Transposition

- A transposed row vector becomes a column (and vice versa)

\[
x = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \quad x^T = \begin{bmatrix} a & b & c \end{bmatrix} \\
\]

- A transposed matrix gets all its row (or column) vectors transposed in order

\[
X = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}, \quad X^T = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix} \\
\]

- MATLAB syntax: \( a' \)
Vector multiplication

- Multiplication by scalar

\[ d[a \ b \ c] = [da \ db \ dc] \]

- Dot product, or inner product
  - Vectors must have the same number of elements
  - Row vector times column vector = scalar

\[ [a \ b \ c] \cdot [d \ e \ f] = a \cdot d + b \cdot e + c \cdot f \]

- Outer product or vector direct product
  - Column vector times row vector = matrix

\[ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} d & e & f \end{bmatrix} = \begin{bmatrix} a \cdot d & a \cdot e & a \cdot f \\ b \cdot d & b \cdot e & b \cdot f \\ c \cdot d & c \cdot e & c \cdot f \end{bmatrix} \]
Vector dot product

• Example:
  – Coordinates are yards, not ave/st
  – \( \mathbf{a} = [200 \ 1600], \mathbf{b} = [770 \ 300] \)

• The dot product of the two vectors relates to the length of a projection
  – How much of the first vector have we covered by following the second one?
  – Must normalize by the length of the “target” vector

\[
\frac{\mathbf{a} \cdot \mathbf{b}^T}{\|\mathbf{a}\|} = \frac{[200 \ 1600] \cdot [770 \ 300]}{\|[200 \ 1600]\|} \approx 393\text{yd}
\]
• Vectors are spectra
  – Energy at a discrete set of frequencies
  – Actually $1 \times 4096$
  – $X$ axis is the \textit{index} of the number in the vector
    • Represents frequency
  – $Y$ axis is the value of the number in the vector
    • Represents magnitude
Vector dot product

- How much of C is also in E
  - How much can you fake a C by playing an E
  - \( \frac{C \cdot E}{|C||E|} = 0.1 \)
  - Not very much
- How much of C is in C2?
  - \( \frac{C \cdot C2}{|C| \cdot |C2|} = 0.5 \)
  - Not bad, you can fake it
- To do this, C, E, and C2 must be the same size
Vector outer product

- The column vector is the spectrum
- The row vector is an amplitude modulation
- The outer product is a spectrogram
  - Shows how the energy in each frequency varies with time
  - The pattern in each column is a scaled version of the spectrum
  - Each row is a scaled version of the modulation
Multiplying a vector by a matrix

• Generalization of vector scaling

\[
\begin{bmatrix}
a \\
b \\
c
\end{bmatrix} \cdot d =
\begin{bmatrix}
ad \\
bd \\
\cd
\end{bmatrix}
\]

– **Left multiplication**: Dot product of each vector pair

\[
A \cdot B =
\begin{bmatrix}
\leftarrow a_1 & \rightarrow \\
\leftarrow a_2 & \rightarrow \\
\end{bmatrix} \cdot \begin{bmatrix}
\uparrow \\
b \\
\downarrow
\end{bmatrix} =
\begin{bmatrix}
a_1 \cdot b \\
a_2 \cdot b
\end{bmatrix}
\]

– Dimensions must match!!

  • No. of columns of matrix = size of vector
  • Result inherits the number of rows from the matrix
Multiplying a vector by a matrix

• Generalization of vector multiplication

\[ d \cdot [a \ b \ c] = [da \ db \ dc] \]

  – **Right multiplication**: Dot product of each vector pair

  \[ A \cdot B = \begin{bmatrix} \uparrow & \uparrow \\ \downarrow & \downarrow \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = [a.b_1 \ a.b_2] \]

  – Dimensions must match!!

  • No. of rows of matrix = size of vector
  • Result inherits the number of columns from the matrix
Multiplication of vector space by matrix

- The matrix rotates and scales the space
  - Including its own vectors

\[
Y = \begin{bmatrix}
0.3 & 0.7 \\
-1.3 & 1.6
\end{bmatrix}
\]
Multiplication of vector space by matrix

• The normals to the row vectors in the matrix become the new axes
  – X axis = normal to the second row vector
  • Scaled by the inverse of the length of the first row vector
Matrix Multiplication

• The k-th axis corresponds to the normal to the hyperplane represented by the 1..k-1,k+1..N-th row vectors in the matrix
  – Any set of K-1 vectors represent a hyperplane of dimension K-1 or less

• The distance along the new axis equals the length of the projection on the k-th row vector
  – Expressed in inverse-lengths of the vector

\[
\begin{bmatrix}
a & d & g \\
b & e & h \\
c & f & i
\end{bmatrix}
\]
Matrix Multiplication: Column space

\[
\begin{bmatrix}
  a & b & c \\
  d & e & f \\
\end{bmatrix}
\begin{bmatrix}
  x \\
  y \\
  z \\
\end{bmatrix}
= x\begin{bmatrix}
  a \\
  d \\
\end{bmatrix} + y\begin{bmatrix}
  b \\
  e \\
\end{bmatrix} + z\begin{bmatrix}
  c \\
  f \\
\end{bmatrix}
\]

• So much for spaces .. what does multiplying a matrix by a vector really do?
• It *mixes* the column vectors of the matrix using the numbers in the vector
• The *column space* of the Matrix is the complete set of all vectors that can be formed by mixing its columns
Matrix Multiplication: Row space

\[
\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = x[a & b & c] + y[d & e & f]
\]

• Left multiplication mixes the row vectors of the matrix.
• The row space of the Matrix is the complete set of all vectors that can be formed by mixing its rows.
Matrix multiplication: Mixing vectors

• A physical example
  – The three column vectors of the matrix X are the spectra of three notes
  – The multiplying column vector Y is just a mixing vector
  – The result is a sound that is the mixture of the three notes
Matrix multiplication: Mixing vectors

- Mixing two images
  - The images are arranged as columns
    - position value not included
  - The result of the multiplication is rearranged as an image

\[
\begin{bmatrix}
75.0 \\
25.0 \\
\end{bmatrix}
\times
\begin{bmatrix}
0.25 \\
0.75 \\
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
0 \\
0 \\
\end{bmatrix}
\]
Multiplying matrices

• Simple vector multiplication: Vector outer product

\[
ab = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \cdot \begin{bmatrix} b_1 & b_2 \end{bmatrix} = \begin{bmatrix} a_1b_1 & a_1b_2 \\ a_2b_1 & a_2b_2 \end{bmatrix}
\]
Multiplying matrices

• Generalization of vector multiplication
  – Outer product of dot products!!
  
  \[ A \cdot B = \left[ \begin{array}{c}
    a_1 \\
    a_2 \\
  \end{array} \right] \cdot \left[ \begin{array}{c}
    b_1 \\
    b_2 \\
  \end{array} \right] = \left[ \begin{array}{c}
    a_1 \cdot b_1 \\
    a_1 \cdot b_2 \\
    a_2 \cdot b_1 \\
    a_2 \cdot b_2 \\
  \end{array} \right] \]

  – Dimensions must match!!
    • Columns of first matrix = rows of second
    • Result inherits the number of rows from the first matrix and the number of columns from the second matrix
Multiplying matrices: Another view

• Simple vector multiplication: Vector inner product

\[
ab = \begin{bmatrix} a_1 & a_2 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = a_1b_1 + a_2b_2
\]
Matrix multiplication: another view

\[ A \cdot B = \begin{bmatrix} a_1 & a_2 \\ \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} \leftarrow & b_1 & \rightarrow \\ \leftarrow & b_2 & \rightarrow \end{bmatrix} = a_2 b_2 + a_2 b_2 \]

- The outer product of the first column of \( A \) and the first row of \( B \) + outer product of the second column of \( A \) and the second row of \( B \) + ....

- **Sum of outer products**
Why is that useful?

- Sounds: Three notes modulated independently

\[
X = \begin{bmatrix}
1 & 3 & 0 \\
. & . & 0 \\
9 & 24 & . \\
. & . & 1
\end{bmatrix}
\]

\[
Y = \begin{bmatrix}
0 & 0.5 & 0.75 & 1 & 0.75 & 0.5 & 0 & \ldots & . \\
1 & 0.9 & 0.7 & 0.5 & 0 & 0.5 & \ldots & . \\
0.5 & 0.6 & 0.7 & 0.8 & 0.9 & 0.95 & 1 & \ldots & . \\
\end{bmatrix}
\]
Matrix multiplication: Mixing modulated spectra

- Sounds: Three notes modulated independently
Matrix multiplication: Mixing modulated spectra

• Sounds: Three notes modulated independently
Matrix multiplication: Mixing modulated spectra

• Sounds: Three notes modulated independently
Matrix multiplication: Mixing modulated spectra

• Sounds: Three notes modulated independently
Matrix multiplication: Mixing modulated spectra

- Sounds: Three notes modulated independently
Matrix multiplication: Image transition

- Image 1 fades out linearly
- Image 2 fades in linearly

\[
\begin{bmatrix}
  i_1 & j_1 \\
  i_2 & j_2 \\
  \vdots & \vdots \\
  \vdots & \vdots \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
  1 & .9 & .8 & .7 & .6 & .5 & .4 & .3 & .2 & .1 & 0 \\
  0 & .1 & .2 & .3 & .4 & .5 & .6 & .7 & .8 & .9 & 1 \\
\end{bmatrix}
\]
Matrix multiplication: Image transition

- Each column is one image
  - The columns represent a sequence of images of decreasing intensity
- Image 1 fades out linearly
Matrix multiplication: Image transition

- Image 2 fades in linearly
Matrix multiplication: Image transition

- Image 1 fades out linearly
- Image 2 fades in linearly
The Identity Matrix

- An identity matrix is a square matrix where
  - All diagonal elements are 1.0
  - All off-diagonal elements are 0.0
- Multiplication by an identity matrix does not change vectors

\[ Y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]
Diagonal Matrix

\[ Y = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \]

- All off-diagonal elements are zero
- Diagonal elements are non-zero
- Scales the axes
  - May flip axes
Diagonal matrix to transform images

• How?
Stretching

Location-based representation

Scaling matrix – only scales the X axis
  – The Y axis and pixel value are scaled by identity

Not a good way of scaling.

\[
\begin{bmatrix}
2 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 2 & 2 & 2 & 10 \\
1 & 2 & 1 & 5 & 6 & 10 \\
1 & 1 & 1 & 0 & 1 & 1
\end{bmatrix}
\]
## Stretching

\[
D = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{bmatrix}
\]

\[
A = \begin{bmatrix}
1 & 0.5 & 0 & 0 & . \\
0 & 0.5 & 1 & 0.5 & . \\
0 & 0 & 0 & 0.5 & . \\
0 & 0 & 0 & 0 & . \\
\end{bmatrix}
\text{(Nx2N)}
\]

\[
\text{Newpic} = DA
\]

- Better way
- **Interpolate**

\[N \text{ is the width of the original image}\]
Modifying color

\[ P = \begin{bmatrix} R \\ G \\ B \end{bmatrix} \]

\[ Newpic = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

- Scale only Green
Permutation Matrix

• A permutation matrix simply rearranges the axes
  – The row entries are axis vectors in a different order
  – The result is a combination of rotations and reflections

• The permutation matrix effectively *permutes* the arrangement of the elements in a vector

\[
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
\end{bmatrix}
= 
\begin{bmatrix}
y \\
z \\
x \\
\end{bmatrix}
\]
Permutation Matrix

- Reflections and 90 degree rotations of images and objects
Permutation Matrix

• Reflections and 90 degree rotations of images and objects
  – Object represented as a matrix of 3-Dimensional “position” vectors
  – Positions identify each point on the surface

\[
P = \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
x_1 & x_2 & \ldots & x_N \\
y_1 & y_2 & \ldots & y_N \\
z_1 & z_2 & \ldots & z_N \\
\end{bmatrix}
\]
Rotation Matrix

\[ x' = x \cos \theta - y \sin \theta \]
\[ y' = x \sin \theta + y \cos \theta \]

\[
R_\theta = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
\]

\[
X = \begin{bmatrix}
x \\
y
\end{bmatrix}
\]

\[
X_{\text{new}} = \begin{bmatrix}
x' \\
y'
\end{bmatrix}
\]

\[
R_\theta X = X_{\text{new}}
\]

• A rotation matrix *rotates* the vector by some angle \( \theta \)

• Alternately viewed, it rotates the axes
  – The new axes are at an angle \( \theta \) to the old one
Rotating a picture

Note the representation: 3-row matrix
- Rotation only applies on the “coordinate” rows
- The value does not change
- Why is pacman grainy?
3-D Rotation

• 2 degrees of freedom
  – 2 separate angles
• What will the rotation matrix be?
Matrix Operations: Properties

- $A + B = B + A$
- $AB \neq BA$
• What would we see if the cone to the left were transparent if we looked at it from above the plane shown by the grid?
  – Normal to the plane
  – Answer: the figure to the right

• How do we get this? Projection
Consider any plane specified by a set of vectors $W_1, W_2, \ldots$
- Or matrix $[W_1 \ W_2 \ldots]$
- Any vector can be projected onto this plane
- The matrix $A$ that rotates and scales the vector so that it becomes its projection is a projection matrix
Given a set of vectors $W_1, W_2$, which form a matrix $W = [W_1 \ W_2 \ldots]$.

The projection matrix to transform a vector $X$ to its projection on the plane is

$$ P = W \ (W^T W)^{-1} \ W^T $$

- We will visit matrix inversion shortly.

Magic – any set of vectors from the same plane that are expressed as a matrix will give you the same projection matrix

$$ P = V \ (V^T V)^{-1} \ V^T $$
Projections

- HOW?
Projections

• Draw any two vectors $W_1$ and $W_2$ that lie on the plane
  – *ANY two so long as they have different angles*
• Compose a matrix $W = [W_1 \ W_2]$
• Compose the projection matrix $P = W (W^T W)^{-1} W^T$
• Multiply every point on the cone by $P$ to get its projection
• View it 😊
  – I’m missing a step here – what is it?
• The projection actually projects it onto the plane, but you’re still seeing the plane in 3D
  – The result of the projection is a 3-D vector
  – \( P = W(W^TW)^{-1}W^T = 3\times3 \), \( P*\text{Vector} = 3\times1 \)
  – The image must be rotated till the plane is in the plane of the paper
    • The Z axis in this case will always be zero and can be ignored
    • How will you rotate it? (remember you know W1 and W2)
Projection matrix properties

• The projection of any vector that is already on the plane is the vector itself
  – \( \text{Px} = x \) if \( x \) is on the plane
  – If the object is already on the plane, there is no further projection to be performed

• The projection of a projection is the projection
  – \( P(\text{Px}) = \text{Px} \)
  – That is because \( \text{Px} \) is already on the plane

• Projection matrices are *idempotent*
  – \( P^2 = P \)
    - Follows from the above
Projections: A more physical meaning

• Let $W_1, W_2 \ldots W_k$ be “bases”

• We want to explain our data in terms of these “bases”
  – We often cannot do so
  – But we can explain a significant portion of it

• The portion of the data that can be expressed in terms of our vectors $W_1, W_2, \ldots W_k$, is the projection of the data on the $W_1 \ldots W_k$ (hyper) plane
  – In our previous example, the “data” were all the points on a cone, and the bases were vectors on the plane
Projection: an example with sounds

- The spectrogram (matrix) of a piece of music

- How much of the above music was composed of the above notes
  - I.e. how much can it be explained by the notes
Projection: one note

- The spectrogram (matrix) of a piece of music

- $\mathbf{M} = \text{spectrogram}; \quad \mathbf{W} = \text{note}$
- $\mathbf{P} = \mathbf{W} (\mathbf{W}^\top \mathbf{W})^{-1} \mathbf{W}^\top$
- Projected Spectrogram $= \mathbf{P} \ast \mathbf{M}$
Projection: one note – cleaned up

The spectrogram (matrix) of a piece of music

- Floored all matrix values below a threshold to zero
Projection: multiple notes

- The spectrogram (matrix) of a piece of music

\[ P = W (W^T W)^{-1} W^T \]

- Projected Spectrogram = \( P \times M \)
Projection: multiple notes, cleaned up

- The spectrogram (matrix) of a piece of music

\[
P = W (W^T W)^{-1} W^T
\]

- Projected Spectrogram = P * M
Projection and Least Squares

• Projection actually computes a *least squared error* estimate

• For each vector \( V \) in the music spectrogram matrix
  – Approximation: \( V_{\text{approx}} = a \cdot \text{note1} + b \cdot \text{note2} + c \cdot \text{note3}.. \)
  
  \[
  V_{\text{approx}} = \begin{bmatrix}
  \text{note1} \\
  \text{note2} \\
  \text{note3}
  \end{bmatrix}
  \begin{bmatrix}
  a \\
  b \\
  c
  \end{bmatrix}
  \]

  – Error vector \( E = V - V_{\text{approx}} \)
  – Squared error energy for \( V \) \( e(V) = \text{norm}(E)^2 \)
  – Total error = sum over all \( V \) \{ \( e(V) \) \} = \( \sum_v e(V) \)

• Projection computes \( V_{\text{approx}} \) for all vectors such that Total error is minimized
  – It does not give you “a”, “b”, “c”.. Though
    • That needs a different operation – the inverse / pseudo inverse
Perspective

- The picture is the equivalent of “painting” the viewed scenery on a glass window
- Feature: The lines connecting any point in the scenery and its projection on the window merge at a common point
  - The eye
  - As a result, parallel lines in the scene *apparently* merge to a point
An aside on Perspective..

- Perspective is the result of convergence of the image to a point
- Convergence can be to multiple points
  - Top Left: One-point perspective
  - Top Right: Two-point perspective
  - Right: Three-point perspective
Representing Perspective

• Perspective was not always understood.
• Carefully represented perspective can create illusions.
Central Projection

The positions on the “window” are scaled along the line.

To compute \((x, y)\) position on the window, we need \(z\) (distance of window from eye), and \((x', y', z')\) (location being projected).

\[
\frac{x}{x'} = \frac{y}{y'} = \frac{z}{z'}
\]

Property of a line through origin

\[
\alpha = \frac{z}{z'} \quad x = \alpha x' \quad y = \alpha y'
\]
Homogeneous Coordinates

- Represent points by a triplet
  - Using yellow window as reference:
    - \((x, y) = (x, y, 1)\)
    - \((x', y') = (x, y, c') \quad c' = \alpha'/\alpha\)
  - Locations on line generally represented as \((x, y, c)\)
    - \(x' = x/c'\, \quad y' = y/c'\)
Homogeneous Coordinates in 3-D

- Points are represented using FOUR coordinates
  - \((X,Y,Z,c)\)
  - “c” is the “scaling” factor that represents the distance of the actual scene
- Actual Cartesian coordinates:
  - \(X_{\text{actual}} = X/c, \ Y_{\text{actual}} = Y/c, \ Z_{\text{actual}} = Z/c\)
Homogeneous Coordinates

- In both cases, constant “c” represents distance along the line with respect to a reference window
  - In 2D the plane in which all points have values (x,y,1)
- Changing the reference plane changes the representation
- I.e. there may be multiple Homogenous representations (x,y,c) that represent the same cartesian point (x’ y’)

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