

Prediction and Search in Probabilistic Worlds

Markov Systems, Markov Decision Processes, and Dynamic Programming

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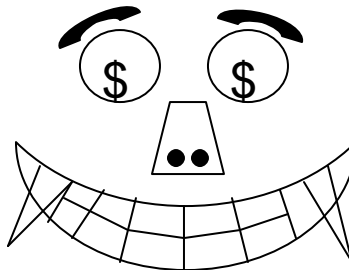
April 21st, 2002

Discounted Rewards

An assistant professor gets paid, say, 20K per year.

How much, in total, will the A.P. earn in their life?

$20 + 20 + 20 + 20 + 20 + \dots = \text{Infinity}$



What's wrong with this argument?

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Discounted Rewards

“A reward (payment) in the future is not worth quite as much as a reward now.”

- Because of chance of obliteration
- Because of inflation

Example:

Being promised \$10,000 next year is worth only 90% as much as receiving \$10,000 right now.

Assuming payment n years in future is worth only $(0.9)^n$ of payment now, what is the AP's **Future Discounted Sum of Rewards** ?

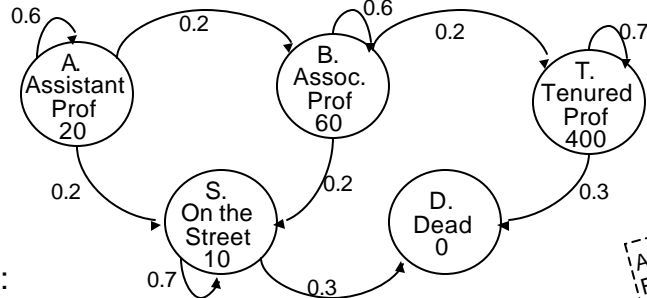
Discount Factors

People in economics and probabilistic decision-making do this all the time.

The “Discounted sum of future rewards” using discount factor γ is

$$\begin{aligned} & (\text{reward now}) + \\ & \gamma (\text{reward in 1 time step}) + \\ & \gamma^2 (\text{reward in 2 time steps}) + \\ & \gamma^3 (\text{reward in 3 time steps}) + \\ & \quad \vdots \\ & \quad \vdots \quad (\text{infinite sum}) \end{aligned}$$

The Academic Life



Define:

J_A = Expected discounted future rewards starting in state A

J_B = Expected discounted future rewards starting in state B

J_T = " " " " " " " T

J_S = " " " " " " " S

J_D = " " " " " " " D

How do we compute J_A, J_B, J_T, J_S, J_D ?

Assume Discount
Factor $\gamma = 0.9$

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Computing the Future Rewards of an Academic

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A Markov System with Rewards...

- Has a set of states $\{S_1 S_2 \dots S_N\}$
- Has a transition probability matrix

$$P = \begin{pmatrix} P_{11} & P_{12} & \dots & P_{1N} \\ P_{21} & & & \\ \vdots & & & \\ P_{N1} & \dots & & P_{NN} \end{pmatrix} \quad P_{ij} = \text{Prob}(\text{Next} = S_j \mid \text{This} = S_i)$$

- Each state has a reward. $\{r_1 r_2 \dots r_N\}$
- There's a discount factor γ . $0 < \gamma < 1$

On Each Time Step ...

0. Assume your state is S_i
1. You get given reward r_i
2. You randomly move to another state
 $P(\text{NextState} = S_j \mid \text{This} = S_i) = P_{ij}$
3. All future rewards are discounted by γ

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Solving a Markov System

Write $J^*(S_i) =$ expected discounted sum of future rewards starting in state S_i

$J^*(S_i) = r_i + \gamma$ (Expected future rewards starting from your next state)

$$= r_i + \gamma(P_{i1}J^*(S_1) + P_{i2}J^*(S_2) + \dots + P_{iN}J^*(S_N))$$

Using vector notation write

$$\underline{J} = \begin{pmatrix} J^*(S_1) \\ J^*(S_2) \\ \vdots \\ J^*(S_N) \end{pmatrix} \quad \underline{R} = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_N \end{pmatrix} \quad \underline{P} = \begin{pmatrix} P_{11} & P_{12} & \dots & P_{1N} \\ P_{21} & & & \\ \vdots & & & \\ P_{N1} & P_{N2} & \dots & P_{NN} \end{pmatrix}$$

Question: can you invent a closed form expression for \underline{J} in terms of \underline{R} , \underline{P} and γ ?

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Solving a Markov System with Matrix Inversion

- Upside: You get an exact answer
- Downside:

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Value Iteration: another way to solve a Markov System

Define $J^1(S_i)$ = Expected discounted sum of rewards over the next 1 time step.

$J^2(S_i)$ = Expected discounted sum rewards during next 2 steps

$J^3(S_i)$ = Expected discounted sum rewards during next 3 steps

:

$J^k(S_i)$ = “ “ “ “ “ “ k steps

$J^1(S_i)$ = (what?)

$J^2(S_i)$ = (what?)

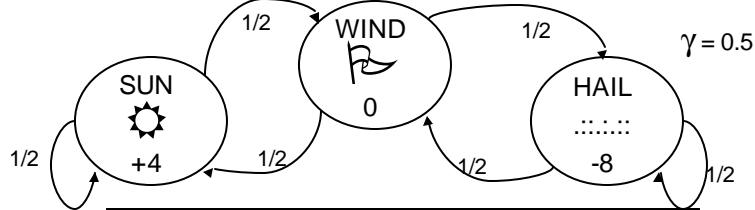
:

$J^{k+1}(S_i)$ = (what?)

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Let's do Value Iteration



| k | $J^k(\text{SUN})$ | $J^k(\text{WIND})$ | $J^k(\text{HAIL})$ |
|---|-------------------|--------------------|--------------------|
| 1 | | | |
| 2 | | | |
| 3 | | | |
| 4 | | | |
| 5 | | | |

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Value Iteration for solving Markov Systems

- Compute $J^1(S_j)$ for each j
- Compute $J^2(S_j)$ for each j

:

- Compute $J^k(S_j)$ for each j

As $k \rightarrow \infty$, $J^k(S_j) \rightarrow J^*(S_j)$. **Why?**

When to stop? When

$$\max_j |J^{k+1}(S_j) - J^k(S_j)| < ?$$

This is faster than matrix inversion (N^3 style)

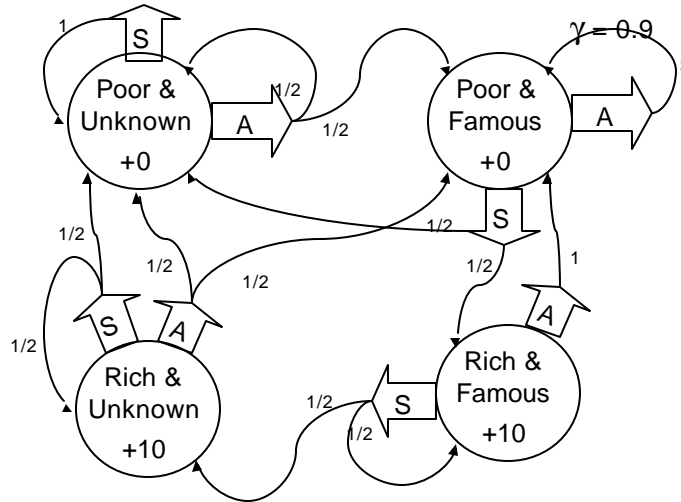
if the transition matrix is sparse

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A Markov Decision Process

You run a startup company.
In every state you must choose between Saving money or Advertising.



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Markov Decision Processes

An MDP has...

- A set of states $\{s_1 \dots s_N\}$
- A set of actions $\{a_1 \dots a_M\}$
- A set of rewards $\{r_1 \dots r_N\}$ (one for each state)
- A transition probability function

$$P_{ij}^k = \text{Prob}(\text{Next} = j | \text{This} = i \text{ and I use action } k)$$

On each step:

0. Call current state S_i
1. Receive reward r_i
2. Choose action $\in \{a_1 \dots a_M\}$
3. If you choose action a_k you'll move to state S_j with probability P_{ij}^k
4. All future rewards are discounted by γ

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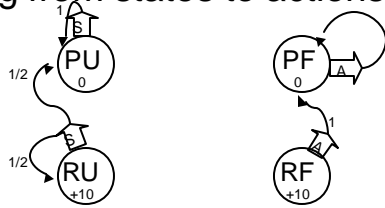
A Policy

A policy is a mapping from states to actions.

Examples

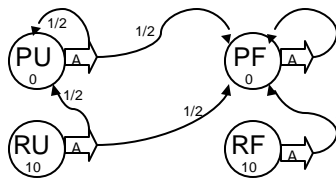
Policy Number 1:

| STATE | ACTION |
|-------|--------|
| PU | S |
| PF | A |
| RU | S |
| RF | A |



Policy Number 2:

| STATE | ACTION |
|-------|--------|
| PU | A |
| PF | A |
| RU | A |
| RF | A |



- How many possible policies in our example?
- Which of the above two policies is best?
- How do you compute the optimal policy?

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Interesting Fact

For every M.D.P. there exists an optimal policy.

It's a policy such that for every possible start state there is no better option than to follow the policy.

(Not proved in this lecture)

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Computing the Optimal Policy

Idea One:

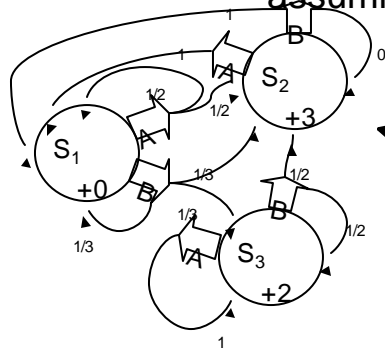
Run through all possible policies.

Select the best.

What's the problem ??

Optimal Value Function

Define $J^*(S_i) =$ Expected Discounted Future Rewards, starting from state S_i , assuming we use the optimal policy



Question

What (by inspection) is an optimal policy for that MDP?

(assume $\gamma = 0.9$)

What is $J^*(S_1)$?

What is $J^*(S_2)$?

What is $J^*(S_3)$?

Computing the Optimal Value Function with Value Iteration

Define

$J^k(S_i)$ = Maximum possible future sum of rewards I can get if I start at state S_i

Note that $J^1(S_i) = r_i$

Let's compute $J^k(S_i)$ for our example

| k | $J^k(\text{PU})$ | $J^k(\text{PF})$ | $J^k(\text{RU})$ | $J^k(\text{RF})$ |
|---|------------------|------------------|------------------|------------------|
| | | | | |
| 1 | | | | |
| 2 | | | | |
| 3 | | | | |
| 4 | | | | |
| 5 | | | | |
| 6 | | | | |

Bellman's Equation

$$J^{n+1}(S_i) = \max_k \left[r_i + \gamma \sum_{j=1}^N P_{ij}^k J^n(S_j) \right]$$

Value Iteration for solving MDPs

- Compute $J^1(S_i)$ for all i
- Compute $J^2(S_i)$ for all i
- :
- Compute $J^k(S_i)$ for all i

.....until converged

$$\left[\text{converged when } \max_i |J^{k+1}(S_i) - J^k(S_i)| < \epsilon \right]$$

...Also known as

Dynamic Programming

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Finding the Optimal Policy

1. Compute $J^*(S_i)$ for all i using Value Iteration (a.k.a. Dynamic Programming)
2. Define the best action in state S_i as

$$\arg \max_k \left[r_i + \gamma \sum_j P_{ij}^k J^*(S_j) \right]$$

(Why?)

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Applications of MDPs

This extends the search algorithms of your first lectures to the case of probabilistic next states.

Many important problems are MDPs....

- ... Robot path planning
- ... Travel route planning
- ... Elevator scheduling
- ... Bank customer retention
- ... Autonomous aircraft navigation
- ... Manufacturing processes
- ... Network switching & routing

Asynchronous D.P.

Value Iteration:

“Backup S_1 ”, “Backup S_2 ”, ..., “Backup S_N ”,
then “Backup S_1 ”, “Backup S_2 ”, ...,
repeat :

: There's no reason that you need to do the
backups in order!

Random Order ...still works. Easy to parallelize (Dyna,
Sutton 91)

On-Policy Order

Simulate the states that the system actually visits.

Efficient Order

e.g. Prioritized Sweeping [Moore 93]

Q-Dyna [Peng & Williams 93]

Another way to compute optimal policies

Policy Iteration

Write $p(S_i)$ = action selected in the i 'th state. Then p is a policy.

Write $p^t = t$ th policy on t th iteration

Algorithm:

p° = Any randomly chosen policy

$\forall i$ compute $J^\circ(S_i)$ = Long term reward starting at S_i using p°

$$p_1(S_i) = \arg \max_a \left[r_i + \gamma \sum_j P_{ij}^a J^\circ(S_j) \right]$$

$J_1 = \dots$

$p_2(S_i) = \dots$

... Keep computing p^1, p^2, p^3, \dots until $p^k = p^{k+1}$. You now have an optimal policy.

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Policy Iteration & Value Iteration: Which is best ???

It depends.

Lots of actions? Choose **Policy Iter**

Already got a fair policy? **Policy Iter**

Few actions, acyclic? **Value Iter**

Best of Both Worlds:

Modified Policy Iteration [Puterman]

...a simple mix of value iteration and policy iteration

3rd Approach

Linear Programming

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Time to Moan

What's the biggest problem(s) with what we've seen so far?

Dealing with large numbers of states

Don't use a Table...

| STATE | VALUE |
|----------------|-------|
| s_1 | |
| s_2 | |
| \vdots | |
| $s_{15122189}$ | |

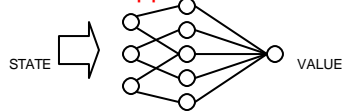
use...

(Generalizers)

Splines



A Function Approximator



(Hierarchies)

Variable Resolution



[Munos 1999]

Multi Resolution



Memory Based



Function approximation for value functions

Polynomials → [Samuel, Boyan, Much O.R. Literature]

Neural Nets → [Barto & Sutton, Tesauro, Crites, Singh, Tsitsiklis]

Backgammon, Pole Balancing, Elevators, Tetris, Cell phones

Checkers, Channel Routing, Radio Therapy

Splines → Economists, Controls

Downside: All convergence guarantees disappear.

Memory-based Value Functions

$J(\text{"state"}) = J(\text{most similar state in memory to "state"})$

or

Average $J(20 \text{ most similar states})$

or

Weighted Average $J(20 \text{ most similar states})$

[Jeff Peng, Atkenson & Schaal,

Geoff Gordon, ← **proved stuff**

Scheider, Boyan & Moore 98]

"Planet Mars Scheduler"

Hierarchical Methods

Continuous State Space: “Split a state when statistically significant that a split would improve performance”

Discrete Space:

Chapman & Kaelbling 92,
McCallum 95 (includes
hidden state)

A kind of Decision
Tree Value Function

Multiresolution

Continuous Space

e.g. Simmons et al 83, Chapman
& Kaelbling 92, Mark Ring 94 ...,
Munos 96

with interpolation!

“Prove needs a higher
resolution”

Moore 93, Moore &
Atkeson 95

A hierarchy with high level “managers” abstracting low level “servants”
Many O.R. Papers, Dayan & Sejnowski’s Feudal learning, Dietterich 1998 (MAX-Q
hierarchy) Moore, Baird & Kaelbling 2000 (airports Hierarchy)

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What You Should Know

- Definition of a Markov System with Discounted rewards
- How to solve it with Matrix Inversion
- How (and why) to solve it with Value Iteration
- Definition of an MDP, and value iteration to solve an MDP
- Policy iteration
- Great respect for the way this formalism generalizes the deterministic searching of the start of the class
- But awareness of what has been sacrificed.

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