Bayesian Networks: Independencies and Inference

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What Independencies does a Bayes Net Model?

- In order for a Bayesian network to model a probability distribution, the following must be true by definition:
  
  Each variable is conditionally independent of all its non-descendants in the graph given the value of all its parents.

- This implies
  
  \[ P(X_1 \ldots X_n) = \prod_{i=1}^{n} P(X_i \mid \text{parents}(X_i)) \]

- But what else does it imply?
What Independencies does a Bayes Net Model?

- Example:
  
  Given $Y$, does learning the value of $Z$ tell us nothing new about $X$?

  I.e., is $P(X|Y, Z)$ equal to $P(X|Y)$?

  Yes. Since we know the value of all of $X$’s parents (namely, $Y$), and $Z$ is not a descendant of $X$, $X$ is conditionally independent of $Z$.

  Also, since independence is symmetric, $P(Z|Y, X) = P(Z|Y)$.

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Quick proof that independence is symmetric

- Assume: $P(X|Y, Z) = P(X|Y)$
- Then:

  $$P(Z|X,Y) = \frac{P(X,Y|Z)P(Z)}{P(X,Y)} \quad \text{(Bayes’s Rule)}$$

  $$= \frac{P(Y|Z)P(X|Y,Z)P(Z)}{P(X|Y)P(Y)} \quad \text{(Chain Rule)}$$

  $$= \frac{P(Y|Z)P(X|Y)P(Z)}{P(X|Y)P(Y)} \quad \text{(By Assumption)}$$

  $$= \frac{P(Y|Z)P(Z)}{P(Y)} = P(Z|Y) \quad \text{(Bayes’s Rule)}$$
What Independencies does a Bayes Net Model?

• Let $I< X, Y, Z >$ represent $X$ and $Z$ being conditionally independent given $Y$.

\[ Y \]

\[ X \]

\[ Z \]

• $I< X, Y, Z >$? Yes, just as in previous example: All $X$’s parents given, and $Z$ is not a descendant.

\[ Z \]

\[ U \]

\[ V \]

\[ X \]

• $I< X, \{ U \}, Z >$? No.

• $I< X, \{ U, V \}, Z >$? Yes.

• Maybe $I< X, S, Z >$ iff $S$ acts a cutset between $X$ and $Z$ in an undirected version of the graph…?
Things get a little more confusing

- X has no parents, so we’re know all its parents’ values trivially
- Z is not a descendant of X
- So, $I_{X,\emptyset, Z}$, even though there’s a undirected path from $X$ to $Z$ through an unknown variable $Y$.
- What if we do know the value of $Y$, though? Or one of its descendants?

The “Burglar Alarm” example

- Your house has a twitchy burglar alarm that is also sometimes triggered by earthquakes.
- Earth arguably doesn’t care whether your house is currently being burgled
- While you are on vacation, one of your neighbors calls and tells you your home’s burglar alarm is ringing. Uh oh!
Things get a lot more confusing

- But now suppose you learn that there was a medium-sized earthquake in your neighborhood. Oh, whew! Probably not a burglar after all.
- Earthquake “explains away” the hypothetical burglar.
- But then it must not be the case that I< Burglar, {Phone Call}, Earthquake>, even though I< Burglar, {}, Earthquake>!

\[ \text{Burglar} \quad \text{Earthquake} \]
\[ \text{Alarm} \]
\[ \text{Phone Call} \]

\[ d\text{-separation} \text{ to the rescue} \]

- Fortunately, there is a relatively simple algorithm for determining whether two variables in a Bayesian network are conditionally independent: \( d\text{-separation} \).

- Definition: \( X \) and \( Z \) are \( d\text{-separated} \) by a set of evidence variables \( E \) iff every undirected path from \( X \) to \( Z \) is “blocked”, where a path is “blocked” iff one or more of the following conditions is true: ...
A path is “blocked” when...

- There exists a variable $V$ on the path such that
  - it is in the evidence set $E$
  - the arcs putting $V$ in the path are “tail-to-tail”

- Or, there exists a variable $V$ on the path such that
  - it is in the evidence set $E$
  - the arcs putting $V$ in the path are “tail-to-head”

- Or, ...

A path is “blocked” when… (the funky case)

- … Or, there exists a variable $V$ on the path such that
  - it is NOT in the evidence set $E$
  - neither are any of its descendants
  - the arcs putting $V$ on the path are “head-to-head”
**d-separation to the rescue, cont’d**

- Theorem [Verma & Pearl, 1998]:
  - If a set of evidence variables $E$ d-separates $X$ and $Z$ in a Bayesian network’s graph, then $I_{X,E,Z}$. 
- $d$-separation can be computed in linear time using a depth-first-search-like algorithm.
- Great! We now have a fast algorithm for automatically inferring whether learning the value of one variable might give us any additional hints about some other variable, given what we already know.
  - “Might”: Variables may actually be independent when they’re not $d$-separated, depending on the actual probabilities involved

**d-separation example**

![Diagram](image)

- $I_{C, \{\}, D}$?
- $I_{C, \{A\}, D}$?
- $I_{C, \{A, B\}, D}$?
- $I_{C, \{A, B, J\}, D}$?
- $I_{C, \{A, B, E, J\}, D}$?
Bayesian Network Inference

- Inference: calculating $P(X|Y)$ for some variables or sets of variables $X$ and $Y$.
- Inference in Bayesian networks is #P-hard!

\[ P(O) \text{ must be } (\#\text{sat. assign.}) \cdot (0.5^{\#\text{inputs}}) \]

Bayesian Network Inference

- **But**...inference is still tractable in some cases.
- Let’s look a special class of networks: trees / forests in which each node has at most one parent.
Decomposing the probabilities

• Suppose we want $P(X_i \mid E)$ where $E$ is some set of evidence variables.

• Let’s split $E$ into two parts:
  • $E_i$ is the part consisting of assignments to variables in the subtree rooted at $X_i$
  • $E_i^+$ is the rest of it

\[
P(\overline{E}_i + \overline{E}_i) = P(X_i \mid E_i, E_i^+) \]

Decomposing the probabilities, cont’d

\[
P(X_i \mid E) = P(X_i \mid E_i^-, E_i^+) = \frac{P(E_i^- \mid X_i, E_i^+) P(X_i \mid E_i^+)}{P(E_i^- \mid E_i^+)} = \bigotimes P(e_i^- \mid x_i) P(x_i \mid e_i^+)
\]
Decomposing the probabilities, cont’d

\[ P(X_i | E) = P(X_i | E_i^-, E_i^+) \]
\[ = \frac{P(E_i^- | X, E_i^+)P(X | E_i^+)}{P(E_i^- | E_i^+)} \]
Decomposing the probabilities, cont’d

\[
P(X_i \mid E) = P(X_i \mid E_i^-, E_i^+) \\
= \frac{P(E_i^- \mid X, E_i^+) P(X \mid E_i^+)}{P(E_i^- \mid E_i^+)} \\
= \frac{P(E_i^- \mid X) P(X \mid E_i^+)}{P(E_i^- \mid E_i^+)} \\
= \alpha \pi(X_i) \lambda(X_i)
\]

Where:

- \(\alpha\) is a constant independent of \(X_i\)
- \(\pi(X_i) = P(X_i \mid E_i^+)\)
- \(\lambda(X_i) = P(E_i^- \mid X_i)\)

Using the decomposition for inference

- We can use this decomposition to do inference as follows. First, compute \(\lambda(X_i) = P(E_i^- \mid X_i)\) for all \(X_i\) recursively, using the leaves of the tree as the base case.
- If \(X_i\) is a leaf:
  - If \(X_i\) is in \(E\): \(\lambda(X_i) = 1\) if \(X_i\) matches \(E\), 0 otherwise
  - If \(X_i\) is not in \(E\): \(E_i^-\) is the null set, so
    \[P(E_i^- \mid X_i) = 1\] (constant)
Quick aside: “Virtual evidence”

- For theoretical simplicity, but without loss of generality, let’s assume that all variables in $E$ (the evidence set) are leaves in the tree.
- Why can we do this WLOG:

\[ \text{Equivalent to} \]

$X_i$ \hspace{1cm} $X_i' \quad \text{Observe } X_i$ \hspace{1cm} $X_i' \quad \text{Observe } X_i'$

Where $P(X_i' \mid X_i) = 1$ if $X_i' = X_i$, 0 otherwise

Calculating $\lambda(X_i)$ for non-leaves

- Suppose $X_i$ has one child, $X_c$.
- Then:

\[ \lambda(X_i) = P(E_i^- \mid X_i) = \]
Calculating $\lambda(X_i)$ for non-leaves

- Suppose $X_i$ has one child, $X_c$.

- Then:

$$\lambda(X_i) = P(E_i^- | X_i) = \sum_j P(E_i^- , X_c = j | X_i)$$

Calculating $\lambda(X_i)$ for non-leaves

- Suppose $X_i$ has one child, $X_c$.

- Then:

$$\lambda(X_i) = P(E_i^- | X_i) = \sum_j P(E_i^- , X_c = j | X_i)$$

$$= \sum_j P(X_c = j | X_i) P(E_i^- | X_i , X_c = j)$$
Calculating $\lambda(X_i)$ for non-leaves

• Suppose $X_i$ has one child, $X_c$.

• Then:

$$
\lambda(X_i) = P(E_i^- | X_i) = \sum_j P(E_i^-, X_c = j | X_i)
= \sum_j P(X_c = j | X_i) P(E_i^- | X_i, X_c = j)
= \sum_j P(X_c = j | X_i) P(E_i^- | X_c = j)
= \sum_j P(X_c = j | X_i) \lambda(X_c = j)
$$

Calculating $\lambda(X_i)$ for non-leaves

• Now, suppose $X_i$ has a set of children, $C$.

• Since $X_i$ $d$-separates each of its subtrees, the contribution of each subtree to $\lambda(X_i)$ is independent:

$$
\lambda(X_i) = P(E_i^- | X_i) = \prod_{X_j \in C} \lambda_j(X_i)
= \prod_{X_j \in C} \left[ \sum_{X_j} P(X_j | X_i) \lambda(X_j) \right]
$$

where $\lambda_j(X_j)$ is the contribution to $P(E_j^- | X_j)$ of the part of the evidence lying in the subtree rooted at one of $X_i$’s children $X_j$. 
We are now $\lambda$-happy

- So now we have a way to recursively compute all the $\lambda(X_i)$’s, starting from the root and using the leaves as the base case.
- If we want, we can think of each node in the network as an autonomous processor that passes a little “$\lambda$ message” to its parent.

The other half of the problem

- Remember, $P(X_i|E) = \alpha \pi(X_i)\lambda(X_i)$. Now that we have all the $\lambda(X_i)$’s, what about the $\pi(X_i)$’s? $\pi(X_i) = P(X_i|E_i^+)$. 
- What about the root of the tree, $X_r$? In that case, $E_r^+$ is the null set, so $\pi(X_r) = P(X_r)$. No sweat. Since we also know $\lambda(X_r)$, we can compute the final $P(X_r)$.
- So for an arbitrary $X_i$ with parent $X_p$, let’s inductively assume we know $\pi(X_p)$ and/or $P(X_p|E)$. How do we get $\pi(X_i)$?
Computing $\pi(X_i)$

$$\pi(X_i) = P(X_i | E_i^+) =$$

Computing $\pi(X_i)$

$$\pi(X_i) = P(X_i | E_i^+) = \sum_j P(X_i, X_j = j | E_i^+)$$
Computing $\pi(X_i)$

$$\pi(X_i) = P(X_i \mid E_i^+) = \sum_j P(X_i, X_p = j \mid E_i^+)$$

$$= \sum_j P(X_i \mid X_p = j, E_i^+) P(X_p = j \mid E_i^+)$$

Computing $\pi(X_i)$

$$\pi(X_i) = P(X_i \mid E_i^+) = \sum_j P(X_i, X_p = j \mid E_i^+)$$

$$= \sum_j P(X_i \mid X_p = j, E_i^+) P(X_p = j \mid E_i^+)$$

$$= \sum_j P(X_i \mid X_p = j) P(X_p = j \mid E_i^+)$$
Computing $\pi(X_i)$

\[
\pi(X_i) = P(X_i \mid E_i^+) = \sum_j P(X_i, X_p = j \mid E_i^+)
\]

\[
= \sum_j P(X_i \mid X_p = j, E_i^+) P(X_p = j \mid E_i^+)
\]

\[
= \sum_j P(X_i \mid X_p = j) P(X_p = j \mid E_i^+)
\]

\[
= \sum_j P(X_i \mid X_p = j) \frac{P(X_p = j \mid E)}{\lambda_i(X_p = j)}
\]

Where $\pi_i(X_p)$ is defined as $\frac{P(X_p \mid E)}{\lambda_i(X_p)}$
We’re done. Yay!

- Thus we can compute all the $\pi(X_i)$’s, and, in turn, all the $P(X_i|E)$’s.
- Can think of nodes as autonomous processors passing $\lambda$ and $\pi$ messages to their neighbors

Conjunctive queries

- What if we want, e.g., $P(A, B | C)$ instead of just marginal distributions $P(A | C)$ and $P(B | C)$?
- Just use chain rule:
  - $P(A, B | C) = P(A | C) \cdot P(B | A, C)$
  - Each of the latter probabilities can be computed using the technique just discussed.
Polymers

• Technique can be generalized to *polytrees*: undirected versions of the graphs are still trees, but nodes can have more than one parent

Dealing with cycles

• Can deal with undirected cycles in graph by
  • clustering variables together

• Conditioning

Set to 0

Set to 1
Join trees

- Arbitrary Bayesian network can be transformed via some evil graph-theoretic magic into a join tree in which a similar method can be employed.

In the worst case the join tree nodes must take on exponentially many combinations of values, but often works well in practice.