Graphical Models and Bayesian Networks

Required reading:

• Ghahramani, section 2, "Learning Dynamic Bayesian Networks" (just 3.5 pages :-)

Optional reading:

• Mitchell, chapter 6.11 Bayesian Belief Networks

Machine Learning 10-701

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Graphical Models

- Key Idea:
  - Conditional independence assumptions useful
  - but Naïve Bayes is extreme!
  - Graphical models express sets of conditional independence assumptions via graph structure
  - Graph structure plus associated parameters define joint probability distribution over set of variables/nodes
- Two types of graphical models:
  - Directed graphs (aka Bayesian Networks)
  - Undirected graphs (aka Markov Random Fields)

today

Graphical Models – Why Care?

- Among most important ML developments of the decade
- Graphical models allow combining:
  - Prior knowledge in form of dependencies/independencies
  - Observed data to estimate parameters
- Principled and ~general methods for
  - Probabilistic inference
  - Learning
- Useful in practice
  - Diagnosis, help systems, text analysis, time series models, ...

Marginal Independence

Definition: X is <u>marginally independent</u> of Y if

$$(\forall i,j) P(X = x_i, Y = y_j) = P(X = x_i) P(Y = y_j)$$

$\underline{P(X,Y)} = P(X)P(Y|X) = P(Y)\underline{P(X|Y)} = \underline{P(X)}P(Y)$

Equivalently, if

$$(\forall i,j) P(X = x_i | Y = y_j) = P(X = x_i)$$

Equivalently, if

$$(\forall i,j) P(Y = y_i | X = x_j) = P(Y = y_i)$$

Conditional Independence

Definition: X is conditionally independent of Y given Z, if the probability distribution governing X is independent of the value of Y, given the value of Z

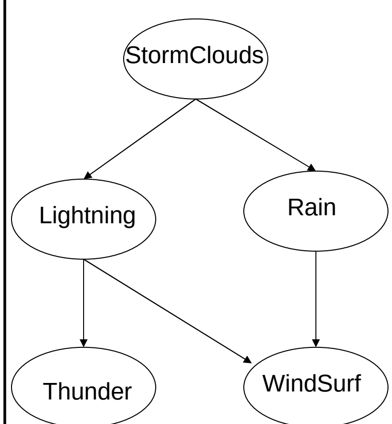
$$(\forall i, j, k) P(X = x_i | Y = y_j, Z = z_k) = P(X = x_i | Z = z_k)$$

Which we often write $P(X|Y, Z) = P(X|Z)$

Marginal indep $P(X|Y) = P(X)$ is special case where $Z = \emptyset$

E.g., $P(Thunder|Rain, Lightning) = P(Thunder|Lightning)$

Bayesian Network



Bayes network: a directed acyclic graph defining a joint probability distribution over a set of variables

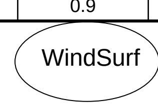
Each node denotes a random variable

Each node is conditionally independent of its non-descendents, given its immediate parents.

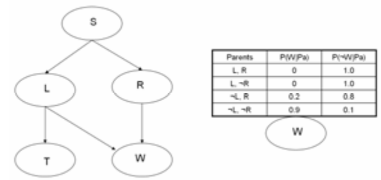
A conditional probability distribution (CPD) is associated with each node N, defining P(N | Parents(N))

$P(W|L,R)$

Parents	P(W|Pa)	P(¬W|Pa)
L, R	0	1.0
L, ¬R	0	1.0
¬L, R	0.2	0.8
¬L, ¬R	0.9	0.1



Bayesian Networks



Parents	P(W|Pa)	P(¬W|Pa)
L, R	0	1.0
L, ¬R	0	1.0
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- Each node denotes a variable
- Edges denote dependencies
- CPD for each node X_i describes $P(X_i \mid Pa(X_i))$
- Joint distribution given by

$$P(X_1 \ldots X_n) = \prod_i P(X_i | Pa(X_i))$$

- Node X_i is conditionally independent of its non-descendents, given its immediate parents

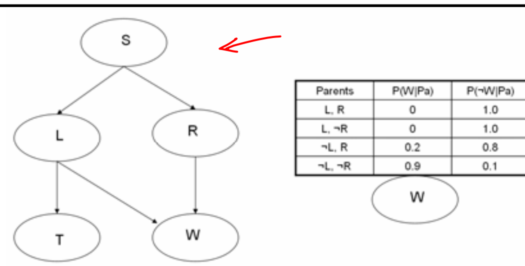
Parents = Pa(X) = immediate parents

Antecedents = parents, parents of parents, ...

Children = immediate children

Descendents = children, children of children, ...

Bayesian Networks



Parents	P(W|Pa)	P(¬W|Pa)
L, R	0	1.0
L, ¬R	0	1.0
¬L, R	0.2	0.8
¬L, ¬R	0.9	0.1

- CPD for each node X_i describes $P(X_i \mid Pa(X_i))$

× cond indep of non-descends
given Pa(x)

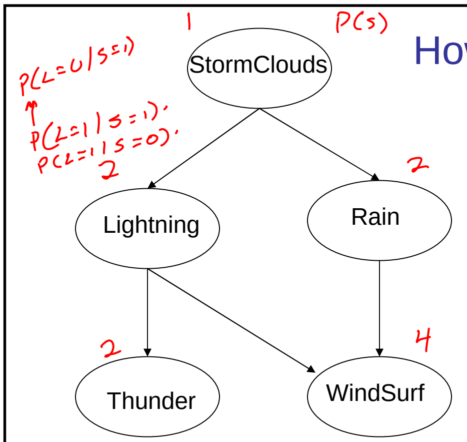
- Chain rule of probability:

$$P(S,L,R,T,W) = P(S)P(L|S)P(R|S,L)P(T|S,L,R)P(W|S,L,R,T)$$

$$= P(S)P(L|S)P(R|S)P(T|L)\ P(W|L\ R)$$

- But in Bayes net:

$$P(X_1 \ldots X_n) = \prod_i P(X_i | Pa(X_i))$$



How Many Parameters?

Parents	P(W|Pa)	P(¬W|Pa)
L, R	0 θ_1	$1-\theta_1$ 1.0
L, ¬R	0 θ_2	$1-\theta_2$ 1.0
¬L, R	0.2 θ_3	0.8
¬L, ¬R	0.9 θ_4	0.1

WindSurf

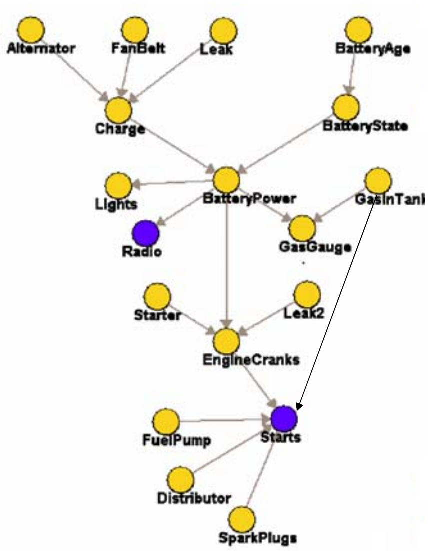
S	L	R	T	W	P( )
0	0	0	0	0	θ_1
0	0	0	0	1	θ_2
⋮					
					θ_{n-1}
					$1-\theta_1-\theta_2 \cdots$

2^5

In full joint distribution? $31 = 2^5 - 1$

Given this Bayes Net? 11

Bayes Net



Inference:

P(BattPower=t | Radio=t, Starts=f)

Most probable explanation:

What is most likely value of Leak, BatteryPower given Starts=f?

Active data collection:

What is most useful variable to observe next, to improve our knowledge of node X?

Algorithm for Constructing Bayes Network

- Choose an ordering over variables, e.g., $X_1, X_2, \ldots X_n$
- For i=1 to n
  - Add X_i to the network
  - Select parents $Pa(X_i)$ as minimal subset of $X_1 \ldots X_{i-1}$ such that

$$P(X_i|Pa(X_i)) = P(X_i|X_1, \ldots, X_{i-1})$$

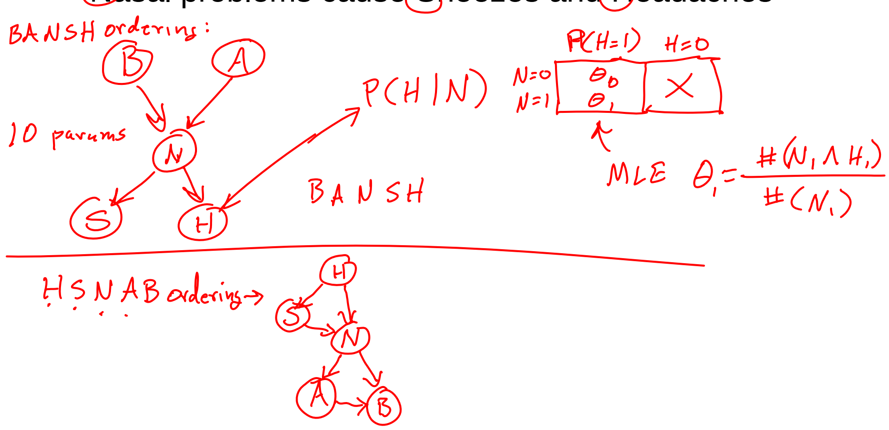
Notice this choice of parents assures

$$P(X_1 \ldots X_n) = \prod_i P(X_i|X_1 \ldots X_{i-1}) \quad \text{(by chain rule)}$$

$$= \prod_i P(X_i|Pa(X_i)) \quad \text{(by construction)}$$

Example

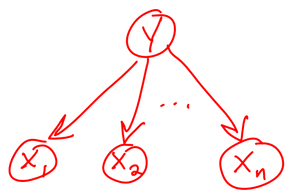
- Bird flu and Allegies both cause Nasal problems
- Nasal problems cause Sneezes and Headaches



What is the Bayes Network for Naïve Bayes?

$f: X \to Y \quad X = \langle X_1, X_2 \cdots X_n \rangle \quad P(Y|X)$

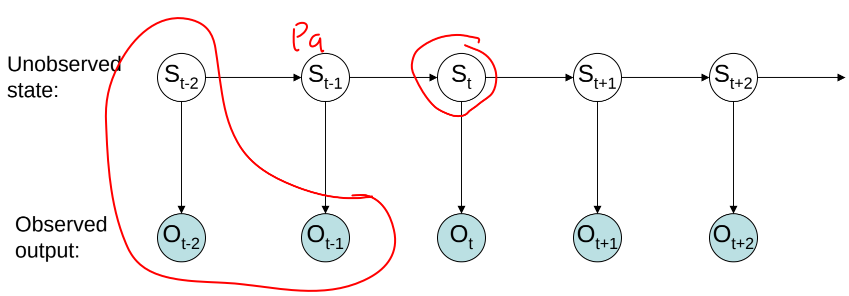
cond indep given Y



Node N c.i. of non-desc given Pa(N)

Bayes Network for a Hidden Markov Model

Assume the future is conditionally independent of the past, given the present

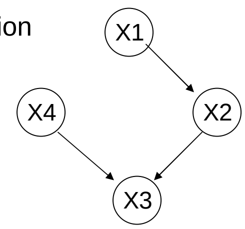


$$P(S_{t-2}, O_{t-2}, S_{t-1}, \ldots, O_{t+2}) = P(S_{t-2})\, P(O_{t-2}|S_{t-2})\, P(S_{t-1}|S_{t-2}) \cdots$$

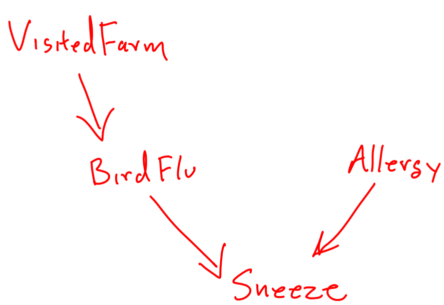
Conditional Independence, Revisited

- We said:
  - Each node is conditionally independent of its non-descendents, given its immediate parents.

- Does this rule give us all of the conditional independence relations implied by the Bayes network?
  - No!
  - E.g., X1 and X4 are conditionally indep given {X2, X3}
  - But X1 and X4 not conditionally indep given X3
  - For this, we need to understand D-separation



Explaining Away



Note BirdFlu $\perp$ Allergy (Marginally indep)

and BirdFlu $\perp$ Allergy given VisitedFarm

but NOT true that BirdFlu $\perp$ Allergy given Sneeze

X and Y are conditionally independent given Z, iff X and Y are D-separated by Z.

D-connection:
If G is a directed graph in which X, Y and Z are disjoint sets of vertices, then X and Y are d-connected by Z in G if and only if there exists an undirected path U between some vertex in X and some vertex in Y such that (1) for every collider C on U, either C or a descendent of C is in Z, and (2) no non-collider on U is in Z.

X and Y are **D-separated** by Z in G if and only if they are not D-connected by Z in G.

Visited Farm

Bird Flu   Allergy

Sneezing

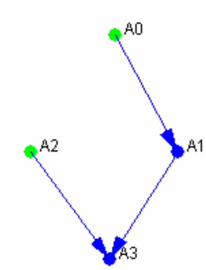
collider

See d-Separation tutorial

http://www.andrew.cmu.edu/user/scheines/tutor/d-sep.html

See d-Separation Applet

http://www.andrew.cmu.edu/user/wimberly/dsep/dSep.html



X, Y	Cond Set	D-Separated	D-Connected
A0 A2	A1 A3	***	
A0 A2	A3		***
A0 A2	A1	***	
A0 A2		***	

A0 and A2 conditionally indep. given {A1, A3}

Inference in Bayes Nets

- In general, intractable (NP-complete)
- For certain cases, tractable
  - Assigning probability to fully observed set of variables
  - Or if just one variable unobserved
  - Or for singly connected graphs (ie., no undirected loops)
    - Belief propagation
- For multiply connected graphs (no directed loops)
    - Junction tree
- Sometimes use Monte Carlo methods
  - Generate a sample according to known distribution
- Variational methods for tractable approximate solutions

Learning in Bayes Nets

- Four categories of learning problems
  - Graph structure may be known/unknown
  - Variables may be observed/unobserved
- Easy case: learn parameters for known graph structure, using fully observed data
- Gruesome case: learn graph and parameters, from partly unobserved data
- More on these in next lectures

Java Bayes Net Applet

http://www.pmr.poli.usp.br/ltd/Software/javabayes/Home/applet.html

What You Should Know

- Bayes nets are convenient representation for encoding dependencies / conditional independence
- BN = Graph plus parameters of CPD's
  - Defines joint distribution over variables
  - Can calculate everything else from that
  - Though inference may be intractable
- Reading conditional independence relations from the graph
  - N cond indep of non-descendents, given parents
  - D-separation
  - 'Explaining away'