15-859(B) Machine Learning Theory

Avrim Blum

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Plan for today:

- problem of "combining expert advice"
- Weighted-majority algorithm
- Generalizations: time-intervals & "sleeping experts"
- Regret-bounds and connections to game-theory (minimax optimality and correlated equilibria)

Mistake-bound model recap

- View learning as a sequence of trials.
- In each trial, algorithm is given x, asked to predict $f(x)$, and then is told correct value.
- Make no assumptions about how examples are chosen.
- Goal is to minimize number of mistakes.

Alg A learns class C with mistake bound M if A makes $\leq$ M mistakes on any sequence of examples consistent with some $f \in C$.

What if there is no perfect function?

Think of as n "experts" giving advice to you. Want to do nearly as well as best of them in hindsight.

- Can view each "expert" as a different $h \in C$.
- Or, think of the special case of C={single variable functions}. Goal is efficient alg that does nearly as well as best single variable.

These are called "regret bounds".
➢Show that our algorithm does nearly as well as best predictor in some large class.

Using "expert" advice

Say we want to predict the stock market.

- We solicit n "experts" for their advice. (Will the market go up or down?)
- We then want to use their advice somehow to make our prediction. E.g.,

Expt 1	Expt 2	Expt 3	neighbor's dog	truth
down	up	up	up	up
down	up	up	down	down
…	…	…	…	…

Can we do nearly as well as best in hindsight?

["expert" ≡ someone with an opinion. Not necessarily someone who knows anything.]

Using "expert" advice

If one expert is perfect, can get $\leq$ lg(n) mistakes with halving alg.

But what if none is perfect? Can we do nearly as well as the best one in hindsight?

Strategy #1:

- Iterated halving algorithm. Same as before, but once we've crossed off all the experts, restart from the beginning.
- Makes at most lg(n)[OPT+1] mistakes, where OPT is #mistakes of the best expert in hindsight.

Seems wasteful. Constantly forgetting what we've "learned". Can we do better?

Weighted Majority Algorithm

Intuition: Making a mistake doesn't completely disqualify an expert. So, instead of crossing off, just lower its weight.

Weighted Majority Alg:

- Start with all experts having weight 1.
- Predict based on weighted majority vote.
- Penalize mistakes by cutting weight in half.

					prediction	correct
weights	1	1	1	1		
predictions	Y	Y	Y	N	Y	Y
weights	1	1	1	.5		
predictions	Y	N	N	Y	N	Y
weights	1	.5	.5	.5		

Analysis: do nearly as well as best expert in hindsight

- M = # mistakes we've made so far.
- m = # mistakes best expert has made so far.
- W = total weight (starts at n).
- After each mistake, W drops by at least 25%. So, after M mistakes, W is at most $n(3/4)^M$.
- Weight of best expert is $(1/2)^m$. So,

$$(1/2)^m \le n(3/4)^M$$
$$(4/3)^M \le n2^m$$
$$M \le 2.4(m + \lg n)$$

constant ratio

Randomized Weighted Majority

2.4(m + lg n) not so good if the best expert makes a mistake 20% of the time. Can we do better? Yes.

- Instead of taking majority vote, use weights as probabilities. (e.g., if 70% on up, 30% on down, then pick 70:30) Idea: smooth out the worst case.
- Also, generalize $\frac{1}{2}$ to 1- ε.

Solves to: $M \le \frac{-m\ln(1-\varepsilon)+\ln(n)}{\varepsilon} \approx (1+\varepsilon/2)m + \frac{1}{\varepsilon}\ln(n)$

M = expected #mistakes

$M \le 1.39m + 2\ln n \leftarrow \varepsilon = 1/2$

$M \le 1.15m + 4\ln n \leftarrow \varepsilon = 1/4$

$M \le 1.07m + 8\ln n \leftarrow \varepsilon = 1/8$

unlike most worst-case bounds, numbers are pretty good.

Analysis

- Say at time t we have fraction F_t of weight on experts that made mistake.
- So, we have probability F_t of making a mistake, and we remove an εF_t fraction of the total weight.
  - $W_{final} = n(1-\varepsilon F_1)(1 - \varepsilon F_2)...$
  - $\ln(W_{final}) = \ln(n) + \sum_t [\ln(1 - \varepsilon F_t)] \le \ln(n) - \varepsilon \sum_t F_t$ (using $\ln(1-x) < -x$)
  - $= \ln(n) - \varepsilon M.$ ($\sum F_t = E$[# mistakes])
- If best expert makes m mistakes, then $\ln(W_{final}) > \ln((1-\varepsilon)^m)$.
- Now solve: $\ln(n) - \varepsilon M > m \ln(1-\varepsilon)$.

$$M \le \frac{-m\ln(1-\varepsilon)+\ln(n)}{\varepsilon} \approx (1+\varepsilon/2)m + \frac{1}{\varepsilon}\log(n)$$

Summarizing

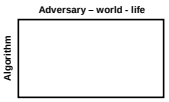
- At most $(1+\varepsilon)$ times worse than best expert in hindsight, with additive $\varepsilon^{-1}\log(n)$.
- Often written in terms of additive loss. If running T time steps, set epsilon to get additive loss $(2T \log n)^{1/2}$
- Define average regret in T time steps as:
  (avg per-day cost of alg) – (avg per-day cost of best fixed expert in hindsight).
  Goes to 0 or better as $T \to \infty$ [= "no-regret" algorithm].

What can we use this for?

- Can use to combine multiple algorithms to do nearly as well as best in hindsight.
- Can apply RWM in situations where experts are making choices that cannot be combined.
  - Choose expert i with probability $p_i = w_i / \sum_i w_i$.
  - E.g., repeated game-playing, repeated route-choosing. (Alg generalizes to case where in each time step, each expert gets a cost in [0,1])

Repeated play of matrix game

- Let's use a no-regret alg.
- Time-average performance guaranteed to approach minimax value V of game (or better, if life isn't adversarial).



- In fact, existence of no-regret algs yields proof of minimax thm….

Using algs for online play

- Rows are "experts". Pick row j with prob w_j/W.
- To keep with terminology, let's talk in terms of gains (doesn't really matter):
  - scale matrix entries to range $[0,1]$.
  - reward expert of gain g by multiplying by $(1+\varepsilon)^g$
  - For any sequence of games, our expected gain
    $$\geq \text{OPT}(1 - \varepsilon/2) - (\ln n)/\varepsilon,$$
    where OPT is best fixed strategy in hindsight (which is at least as good as minimax optimal).
- Claim: this is a proof of the Minimax theorem!

Why?

- What would it mean for minimax to be false?
  - If we know opponents randomized strategy, we can get expected gain $\geq V$, but if we have to choose our randomized strategy first, then opponent can force us to get $\leq V - \delta$.
- This contradicts our bound if we use $\varepsilon = \delta$. Our gain per game is approaching $\text{OPT}(1-\varepsilon/2)$, where $\text{OPT} \geq V$.
- In other words: if there was a gap (V versus V - δ), then for *any* randomized strategy we chose, an opponent knowing our strategy could force us to get no more than V - δ on average per play.
- But, we are doing better.

A natural generalization

- A natural generalization of this setting: say we have a list of n prediction rules, but not all rules fire on any given example.
- E.g., document classification. Rule: "if <word-X> appears then predict <Y>". E.g., if has football then classify as sports.
- E.g., path-planning: "on snowy days, use this route".
- Natural goal: simultaneously, for each rule i, guarantee to do nearly as well as it *on the time steps in which it fires*.
  - For all i, want $E[\text{cost}_i(\text{alg})] \leq (1+\varepsilon)\text{cost}_i(i) + O(\varepsilon^{-1}\log n)$.
- So, if 80% of documents with football *are* about sports, we should have error $\leq$ 21% on them.

"Specialists" or "sleeping experts" problem.

A natural generalization

Generalized version of randomized WM:

- Initialize all rules to have weight 1.
- At each time step, of the rules i that fire, select one with probability $p_i \propto w_i$.
- Update weights:
  - If didn't fire, leave weight alone.
  - If did fire, raise or lower depending on performance compared to weighted average:
    - $R_i = [\sum_j p_j \text{cost}(j)]/(1+\varepsilon) - \text{cost}(i)$
    - $w_i \leftarrow w_i(1+\varepsilon)^{R_i}$
- So, if rule i does exactly as well as weighted average, its weight drops a little. Weight increases if does better than weighted average by more than a $(1+\varepsilon)$ factor.
- Can then prove that total sum of weights never goes up.
- Can extend to rules that can be fractionally on too.

Why does this work?

- Update weights:
  - If didn't fire, leave weight alone.
  - If did fire, raise or lower depending on performance compared to weighted average:
    - $R_i = [\sum_j p_j \text{cost}(j)]/(1+\varepsilon) - \text{cost}(i)$
    - $w_i \leftarrow w_i(1+\varepsilon)^{R_i}$
- Can then prove that total sum of weights never goes up.

- One way to look at weights:
  - $w_i = (1+\varepsilon)^{E[\text{cost}_i(\text{alg})]/(1+\varepsilon) - \text{cost}_i(i)}$
  - I.e., we are explicitly giving large weights to rules for which we have large regret.
  - Since sum of weights $\leq$ n, exponent must be $\leq \log_{1+\varepsilon} n$
- Can extend to rules that can "partially fire" too.

More general forms of regret

1. "best expert" or "external" regret:
   - Given n strategies. Compete with best of them in hindsight.
2. "sleeping expert" or "regret with time-intervals":
   - Given n strategies, k properties. Let S_i be set of days satisfying property i (might overlap). Want to simultaneously achieve low regret over each S_i.
3. "internal" or "swap" regret: like (2), except that S_i = set of days in which we chose strategy i.

Internal/swap-regret

- E.g., each day we pick one stock to buy shares in.
  - Don't want to have regret of the form "every time I bought IBM, I should have bought Microsoft instead".
- Real motivation: connection to correlated equilibria.
  - Distribution over entries in matrix, such that if a trusted party chooses one at random and tells you your part, you have no incentive to deviate.
  - E.g., Shapley game.

Internal/swap-regret, contd

- If all parties run a low internal/swap regret algorithm, then empirical distribution of play is an apx correlated equilibrium.
  - Correlator chooses random time $t \in \{1,2,\ldots,T\}$. Tells each player to play the action j they played in time t (but does not reveal value of t).
  - Expected incentive to deviate: $\sum_j \Pr(j)(\text{Regret}|j)$ = swap-regret of algorithm
  - So, this gives a nice distributed way to get apx correlated equilibria in multiplayer games.

Internal/swap-regret, contd

Algorithms for achieving low regret of this form:

- Foster & Vohra, Hart & Mas-Colell, Fudenberg & Levine.
- Can also convert any "best expert" algorithm into one achieving low swap regret.
- Unfortunately, time to achieve ε regret is linear in n rather than log(n)….