

Fair division, Part 2

Herve Moulin, *Rice University*

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cost and surplus sharing

full responsibility for output demands (or input contributions) $x_i, i \in N$

externalities in production $\rightarrow C(x_i, i \in N)$ cost (or surplus) to share

function C is known to system designer

fair division determined by counterfactuals = costs at alternative demand profiles

examples: connectivity demands (mcst, edge cover, vertex cover), exploiting a commons

goal #1: axiomatic analysis of fairness

→ realistic for **inelastic demands** (no private information)

well developed for:

- binary demands: $x_i \in \{0, 1\} \rightarrow \{N \supseteq S \rightarrow c(S)\}$ TU-cooperative game
- one dimensional demands: $x_i \in \mathbb{N}$ or \mathbb{R}_+ , $C(\sum_N x_i)$ or $C(x_i, i \in N)$, monotone

→ here we discuss only the former (surveys for the latter: [29], [18])

we can extract from most problems the canonical Stand Alone TU game:

$$c(S) = C(x_i, i \in S; 0, j \in N \setminus S)$$

and can ignore any other information to derive cost shares

this *reductionist* approach is hard to justify when other info is available

goal #2: efficient usage of the “commons” C

key simplifying assumption: preferences quasilinear in money \leftrightarrow utility measures willingness to pay

\Rightarrow efficiency means maximizing total utility, and performing arbitrary cash transfers

\Rightarrow efficient surplus (or social cost, [26], see below) can be compared to surplus at equilibrium outcome of the mechanism

\rightarrow ordinal preferences preclude such cardinal measurements

private information on utilities \Rightarrow elicited by playing a well designed mechanism

equilibrium of mechanism should

→ divide cost responsibilities fairly

→ achieve efficient or near-efficient outcome

two closely related forms of mechanisms:

- demand game: agent i sends demand, mechanism returns cost share
- direct revelation mechanism: agent i reports utility, mechanism returns allocation = output and cost share

→ a demand game with a single equilibrium defines a revelation mechanism

→ under Consumer Sovereignty, a strategyproof (SP) mechanism generates a cost sharing rule, hence a demand game

Group Strategyproofness, and Weak Group Strategyproofness are within reach

simple WGSP mechanism: agents pay their incremental costs along a **fixed priority** ordering

⇒ the Random Priority mechanism (RP) is fair and universally SP (no restriction on risk attitude)

how inefficient is RP?

what SP, GSP, or WGSP mechanisms are more fair ?

→ some partial answers to both questions

we discuss fairness issues in section 1, incentives and mechanism design in sections 2,3,4

1 TU cooperative games

elementary surveys: [1], [14]

(N, v)

$N \ni i$: agents, $|N| = n$

$v : 2^N \setminus \emptyset \ni S \rightarrow v(S) \in \mathbb{R}_+$ the value function

a.k.a. the Stand Alone cost (or surplus) of coalition S

→ individual property rights determine Stand Alone surplus/cost opportunities for all coalitions (subsets of agents)

real property rights lead to core stable outcomes, must be curtailed if the core is empty

virtual property rights define the Stand Alone tests (individual and coalitional), competing with other fairness requirements

in the applications the virtual interpretation dominates

→ in submodular cost sharing, the SA cost of serving S only is an *upper* bound derived from real property rights of S , or the virtual right of refusing to subsidize $N \setminus S$

→ in supermodular cost sharing, the SA cost of serving S only is a *lower* bound derived from the virtual right of sole access to the commons

important properties of a game (N, v)

monotonicity: $S \subset S' \Rightarrow v(S) \leq v(S')$

super/sub-additivity: $S \cap S' = \emptyset \Rightarrow v(S) + v(S') \leq v(S \cup S')$ (resp. \geq)

super/sub-modularity: $v(S) + v(S') \leq v(S \cup S') + v(S \cap S')$ (resp. \geq)

allocation: a division x of $v(N)$ among N

$$x \in \mathbb{R}^N \text{ and } x_N = v(N)$$

- when v is superadditive, allocation x meets the *upper-Stand Alone test* (SA_t), if $x_i \geq v(\{i\})$ for all i ; it is in the *upper-Stand Alone Core* (SA_C): if $x_S \geq v(S)$ for all S
- when v is subadditive, allocation x meets the *lower-Stand Alone test* (SA_t), if $x_i \leq v(\{i\})$ for all i ; it is in the *lower-Stand Alone Core* (SA_C): if $x_S \geq v(S)$ for all S

alternative expression of the upper-SAC

$$x_S \leq v(N) - v(N \setminus S) \text{ for all } S$$

the share of S not larger than its “best” marginal contribution, after $N \setminus S$

ditto for the lower-SAC in cost terminology

$$x_S \geq c(N) - c(N \setminus S) \text{ for all } S$$

cost charged to S is at least its “best” marginal cost, after $N \setminus S$; otherwise $N \setminus S$ *subsidizes* S

the lower-SACore (resp. upper-SACore) may be empty when v is superadditive (resp. subadditive)

solution φ (aka *value*): selects an allocation for any game

$$(N, v) \rightarrow \varphi(N, v) = x$$

properties of the solution φ

Stand Alone test (SA_t): $\varphi(N, v)$ meets the lower-SA_ttest whenever v is superadditive, and the upper-SA_ttest when v is subadditive

Stand Alone Core (SAC): $\varphi(N, v)$ is in the lower-SACore(N, v) or in the upper-SACore(N, v) whenever either one is non empty

Coalitional Monotonicity (CM):

$$\{v(S_0) < v'(S_0) \text{ and } v(S) = v'(S) \text{ for all } S \neq S_0\} \Rightarrow x_i \leq x'_i \text{ for all } i \in S_0$$

Proposition ([29]): *SAC and CM are not compatible*

proof by a simple counterexample with five agents

incentive interpretation: for any solution, in some game some coalition can object by standing alone, or some player benefits by sabotaging an innovation

the *marginal contribution* allocation for an ordering i_1, i_2, \dots , of the agents:

$$x_{i_1} = v(\{i_1\}), \quad x_{i_2} = v(\{i_1, i_2\}) - v(\{i_1\}), \dots$$

the Shapley value is the average (expectation) of all marginal contribution vectors

$$x_i = \sum_{0 \leq s \leq n-1} \frac{s!(n-s-1)!}{n!} \sum_{S \subseteq N \setminus i, |S|=s} \{v(S \cup \{i\}) - v(S)\}$$

Shapley characterized his solution by the combination of Anonymity, Additivity in costs, and the Dummy axiom

many alternative characterizations followed, vindicating the star status of this solution for TU cooperative games

→ the Shapley value meets the SAtest and CM, but not the SACore in general

however

Proposition: *if v is super/submodular, the SACore is the convex hull of the marginal contribution vectors (a characteristic property of super/submodularity)*

Corollary *if v is super/submodular the Shapley value is the “center” of the SACore*

Proposition: *if v is super/submodular, the SACore admits a Lorenz dominant selection, called the Dutta Ray solution ([7]).*

for this class of games, the DuttaRay solution is "welfarist under core constraints"

an important solution when the SACore expresses feasibility constraints

fix a game (N, v) and a solution φ defined for all *subgames* (S, v^S) : $S \subseteq N$
and $v^S(T) = v(T)$ for all $T \subseteq S$

Population Monotonicity (PM^+) (resp. PM^-):

$$\{S \subseteq N \setminus j \text{ and } i \in S\} \Rightarrow \varphi_i(S, v^S) \leq \varphi_i(S \cup \{j\}, v^{S \cup \{j\}}) \text{ (resp. } \geq \text{)}$$

PM^+ means that adding a new agent increases (weakly) the shares of existing ones; PM^- means it decreases them (weakly)

both properties are of central interest in strategic cost sharing

Proposition ([28], [7]): *the Shapley value as well as the Dutta Ray solutions are PM^+ (resp. PM^-) in supermodular (resp. submodular) games*

1.1 Application 1: Connectivity games

recall: *inelastic binary demands*

1.1.1 connections on a fixed tree

the simplest case

tree Γ , nodes V

each agent i needs to connect every pair v, v' in a subset of nodes $A_i \subseteq V$

cost of edge e is c_e ; additive

the subset B_i of edges necessary to serve agent i is well defined

Stand Alone costs: $c(S) = \sum_{e \in \cup_S B_i} c_e$

→ submodular

the Shapley value divides equally each c_e between all agents i who need e :
 $e \in B_i$

example: airport landing fees

1.1.2 minimal cost spanning tree (mcst)

each agent i needs to connect one node v_i to the source ω (a special node)

all non-source nodes are occupied

→ the mc (minimal cost, efficient) tree is easy to compute: Prim's and Kruskal's algorithms

Stand Alone cost (public access to all edges): $c(S)$ is cheapest cost of connecting all $v_i, i \in S$, to the source, possibly going through vertices occupied by $N \setminus S$

→ **not** easy to compute (Steiner nodes)

the SA game has a non empty core, but is **not supermodular**

proof: the Bird solution ([2]) is in the core: each agent pays his downstream edge on a mcst

the Bird solution is not a fair division of costs

- discontinuous in costs
- not cost monotonic ($c \leq c' \Rightarrow x \leq x'$)
- not population monotonic ($x_j(N \setminus i) \geq x_j(N)$)

the Shapley value of the SA game is continuous in costs

→ but not in the core

→ nor cost monotonic

→ nor population monotonic

the irreducible cost c^* obtains by the largest cost reduction ($c^* \leq c$) preserving the efficient cost

$$c_e^* = \min_{\gamma(e)} \{ \max_{e' \in \gamma(e)} c_{e'} \}$$

where $\gamma(e)$ is the set of paths connecting the end-nodes of e

→ the c^* -SA game is submodular

its Shapley value is called the *folk solution* ([3], [4], [6], [8], [24])

→ the folk solution is easy to compute ([3])

fix $i \in N$ and order the $n - 1$ costs $c_{ij}^*, j \in N \setminus i$, increasingly

$$\gamma_1 = c_{ij_1}^* \leq \gamma_2 = c_{ij_2}^* \leq \cdots \leq \gamma_{n-1} = c_{ij_{n-1}}^*$$

$$x_i = \frac{1}{n}c_{i\omega}^* + \sum_{k=1}^{n-1} \frac{1}{k(k+1)} \min\{\gamma_k, c_{i\omega}^*\}$$

→ the folk solution is a continuous core selection

→ it is cost and population monotonic

Theorem: *the folk solution is the only symmetric selection of the SA core satisfying **piecewise linearity**: cost shares are additive in edge costs c_e as long as the relative ordering of the edge costs does not change*

1.1.3 minimal cost Steiner tree

same as mcst except that not all nodes are occupied by an agent

⇒ the computation of the efficient cost (and, as before, coalitional SA costs) is hard, can only be approximated

easy 2-approximation

→ the SACore may be empty ([27])

there is no single largest cost reduction preserving the efficient cost

open question: develop the axiomatics of a fair solution in the approximation world

1.1.4 a general problem

sharing public items

$$A \ni a \rightarrow c_a$$

$$N \ni i \rightarrow \mathcal{A}_i \subseteq 2^A$$

agent i is *served* if at least one subset of items $A_i \in \mathcal{A}_i$ is provided

$$c(S) = \min\{c_B \mid \forall i \exists A_i \in \mathcal{A}_i : A_i \subseteq B\}$$

examples: multi-connectivity, connectivity in fixed graphs with cycles, edge cover, set cover, vertex cover, \dots

the SA Core may be empty

the SACore may be too generous to a flexible agent:

$$A = \{a, b\}, N = \{1, 2, 3\}, \mathcal{A}_1 = \{\{a\}\}, \mathcal{A}_2 = \{\{b\}\}, \mathcal{A}_3 = \{\{a\}, \{b\}\}$$

→ in the SA core agent 3 pays nothing:

$$x_1 + x_3 \leq c(\{1, 3\}) = c_a \Rightarrow x_2 \geq c_b; \text{ similarly } x_1 \geq c_a$$

open question: develop an (or several) axiomatically fair division rule(s)
for the public items problem

1.2 Application 2: Division of manna with cash transfers

$N \ni i$: agents, $|N| = n$

$\omega \in \mathbb{R}_+^K$: resources to divide in n shares z^i (divisible commodities)

agent i 's utility is **quasi-linear**: $u_i(z^i) + t^i$

u_i is continuous and monotone

t^i is a cash transfer to i

feasible allocations: $\sum_N z^i = \omega$ and $\sum_N t^i = 0$

efficient allocations: maximize aggregate utility

$$v(N, \omega) = \max\{\sum_N u_i(z^i) \mid \sum_N z^i = \omega\}$$

division rule: $(N, \omega, u_i) \rightarrow (z^i, t^i; i \in N) \rightarrow U_i$

canonical examples: adapt CEEI and ω -EE

→ **new fairness test:** an upper bound on welfare

upper-SACore (u-SAC): $U_S \leq v(S, \omega)$ for all $S \subset N$

right to consume rather than right to extract surplus

→ always feasible (no convexity needed): "utilitarian" solution

u-SAC is incompatible with No Envy:

$$K = 1, \omega = 1, u_1(z) = 5z, u_2(z) = 4z, u_3(z) = z$$

$$\text{EFF} \cap \text{NE}: z^1 = 1, t^2 = t^3 = t, t \geq 4 - 2t \Rightarrow U_3 = t \geq \frac{4}{3} > v(\{3\}, \omega)$$

→ fails also for ω -EE

$$K = 1, \omega = 1, u_1(z) = 4z, u_2(z) = u_3(z) = z$$

$$\omega\text{-EE}: U_1 = 4\lambda, U_2 = U_3 = \lambda \Rightarrow \lambda = \frac{2}{3} \Rightarrow U_{1,2} = \frac{4}{3} > v(\{1, 2\}, \omega)$$

divisible goods and *concave domain*: u_i concave for all i

ULB: $U_i \geq u_i(\frac{1}{n}\omega)$; weakULB: $U_i \geq \frac{1}{n}u_i(\omega)$

impossibility results:

- $\text{EFF} \cap \text{NE} \cap \text{u-SAC} = \emptyset$
- $\text{EFF} \cap \text{weakULB} \cap \text{u-SAC} = \emptyset$
- $\text{EFF} \cap \text{RM} \cap \text{u-SAC} = \emptyset$
- $\text{EFF} \cap \text{PM} = \emptyset$

note: PM strengthens u-SAC

subdomain of the concave domain:

substitutable goods: $\frac{\partial^2 u_i}{\partial z_k^i \partial z_k^i} \leq 0$ for all $1 \leq k, k' \leq K$

Theorem ([15]): *under substitutability, the Shapley value of the SA game meets u -SAC, ULB, RM, and PM*

1.3 Application 3: assignment with money

→ indivisible version of the divisible manna problem

$N \ni i$: agents, $|N| = n$

$A \ni a$: indivisible objects to assign among agents

assignment: $N \ni i \rightarrow a(i) \in A \cup \{\emptyset\}$, one-to-one in A

agent i wants at most one object, utility $u_{ia} \geq 0$

cash transfers: $\sum_N t^i = 0$

efficient assignment a^* : $v(N) = \sum_N u_{ia^*(i)} = \max_a \sum_N u_{ia(i)}$

unanimity utility $una(u_i, A) = \frac{1}{n} \max_a \sum_{j \in N} u_{ia(j)}$

ULB: $U_i \geq una(u_i, A)$

other axioms: NE, RM, u-SAC, PM: identical definitions

CEEI: find a price p such that $u_{ia^*(i)} - p_{a^*(i)} \geq u_{ia} - p_a$ for all i and a

then $U_i = u_{ia^*(i)} - p_{a^*(i)} + \frac{1}{n} \sum_N p_{a^*(i)}$

→ *an allocation is non envious if and only if it is a CEEI allocation*

→ all such allocations meet the ULB

all selections fail RM, PM

ω -EE must be adapted to fit the ULB:

$$U_i = \frac{una(u_i, A)}{\sum_{j \in N} una(u_j, A)} v(N)$$

neither RM nor PM

all the impossibility results for divisible manna still valid

the (adjusted) substitutability condition for divisible manna holds true

\Rightarrow the Shapley value meets u-SAC, ULB, RM, and PM

example: one good, $u_1 \geq u_2 \geq \dots \geq u_n$

CEEI: agent 1 pays t , $\frac{u_2}{n} \leq t \leq \frac{u_1}{n}$ to everyone else

Shapley: $U_n = \frac{u_n}{n}$, $U_{n-1} = \frac{u_n}{n} + \frac{u_{n-1} - u_n}{n-1}$, \dots , $U_1 = \sum_{j=1}^n \frac{u_j - u_{j+1}}{j}$

2 Production games: supermodular costs

general model of the commons with **elastic demands**

utility $u_i(x_i)$: willingness to pay for allocation x_i

cost function $C(x)$

efficiency: to maximize aggregate surplus $\sum_N u_i(x_i) - C(\sum_N x_i)$

Stand Alone surplus: $\max\{\sum_S u_i(x_i) - C(\sum_S x_i)\}$

subadditive but not necessarily submodular

2.1 binary demands, symmetric imc costs

each agent wants (at most) one unit : example scheduling

1-dimensional “type”: willingness to pay $u_1 \geq u_2 \geq \dots$

all units are identical “service”; marginal cost increases: $c_1 \leq c_2 \leq \dots$

→ the *SA surplus* game is submodular

(not true for multi-units demands)

we compare two simple demand games/mechanisms

the **Average Cost** (AC) mechanism

each agent chooses *in* or *out*; if q agents are “in”

$$U_i = u_i - \frac{C(q)}{q} \text{ if } i \text{ is in; } U_i = 0 \text{ if } i \text{ is out}$$

demand function: $d(p) = |\{i | u_i \geq p\}|$

equilibrium quantity of the AC game: q_{ac} solves $d(AC(q)) = q$

\Rightarrow overproduction: $q_e < q_{ac}$

normative properties in equilibrium

→ a Nash equilibrium allocation can generate Envy

→ the demand game may have a strong Battle of the Sexes flavor

→ not SP except in a limit sense

the **Random Priority** (RP) mechanism

law of large numbers \Rightarrow computations easy in the continuous limit case,
e.g., $RP \Leftrightarrow PS$

assume agents maximize their expected utility

Nash equilibrium quantity: q_{rp} solves

$$\int_0^{q_{rp}} \frac{1}{d(C'(t))} dt = 1 \text{ and } C'(q_{rp}) \leq \bar{p} = d^{-1}(0); \text{ or } C'(q_{rp}) = \bar{p}$$

→ overproduction at most 100%: $q_e < q_{rp} \leq 2q_e$ (for any imc C)

normative properties in equilibrium

→ each agent $p \geq C'(0)$ gets positive surplus (service with some probability), while in AC all agents $p \leq d^{-1}(q_{ac})$ get nothing

→ the equilibrium allocation is Pareto inferior to the Shapley allocation \Rightarrow meets the upper-SACore

→ the equilibrium allocation is Non Envious

→ strategyproof revelation mechanism

comparing AC and RP ([5])

→ for quadratic costs and linear demands, RP collects at least 50% of the efficient surplus; RP collects more surplus and overproduces less than AC

→ RP allocation may even Pareto dominate AC allocation: e.g. flat demand

→ AC allocation may not Pareto dominate RP, but may collect larger surplus and overproduce less: e.g. $\frac{1}{d}$ concave

→ the worst absolute surplus loss of RP is smaller than that of AC for any C ; both losses are of the same order if the cost is polynomial: ([11])

open question: is the worst absolute loss of RP optimal among all SP mechanisms?

open question: the structure of strategyproof and budget balanced revelation mechanisms (already hard for $c_1 = 0 < \infty = c_2$!)

2.2 multi-units demands, homogenous imc costs

agent i demands $x_i \in \mathbb{N}$ (discrete model) or \mathbb{R}_+ (continuous model)

utility $u_i(x_i) - y_i$ is concave

cost function: $C(x) = C(\sum_N x_i) = \sum_N y_i$

$C(0) = 0$, C is increasing and convex

a cost sharing rule is $\varphi : x, C \rightarrow y = \varphi(x)$ s.t. $\sum_N y_i = C(\sum_N x_i)$

→ for any profile of utilities $(u_i, i \in N)$ it defines a *demand game*

→ if the demand game has a unique equilibrium (of any kind), this defines a direct revelation mechanism

we look for sharing rules φ generating good incentive properties in the demand game and the revelation mechanism

among those, we look for rules that are fair as well

discrete model

incremental mechanisms (deterministic)

fix a sequence $\mathbb{N} \ni t \rightarrow i(t) \in N$ such that $|\{t | i(t) = j\}| = \infty$ for all j

offer units at successive costs c_1, c_2, \dots , in the order of the sequence

an agent is out after first refusal

Theorem ([17])

1). *The resulting demand game is strictly dominance solvable, its equilibrium is strong, and a coalitional Stackelberg equilibrium; the corresponding revelation mechanism is GSP;*

2). *These capture all GSP mechanisms meeting*

No Charge for No Demand: $x_i = 0 \Rightarrow y_i = 0$

Consumer Sovereignty: for any $k = 0, 1, \dots$, agent i can ensure $x_i = k$

Continuity of cost shares w.r.t. costs c_i .

continuous model

choose a round-robin sequence $\{1, 2, \dots, n, 1, 2, \dots\}$ and an increment δ offered successively at prices $C(\delta), C(2\delta) - C(\delta), C(3\delta) - C(2\delta), \dots$

\Rightarrow same incentives properties

in the limit as $\delta \rightarrow 0$ the **serial cost sharing** rule obtains

if $x_1 \leq x_2 \leq \dots \leq x_n$ the shares are

$$y_1 = \frac{1}{n}C(nx_1); y_2 = y_1 + \frac{1}{n-1}\{C(x_1 + (n-1)x_2) - C(nx_1)\}; \dots$$

$$y_{k+1} = y_k + \frac{1}{n-k}\{C(x_1, \dots, k + (n-k)x_{k+1}) - C(x_1, \dots, k-1 + (n-k+1)x_k)\}$$

$$y_n = y_{n-1} + \{C(x_N) - C(x_{N \setminus n} + x_{n-1})\}$$

incentives properties of the serial demand game/revelation mechanism

Theorem ([21]) *Fix a strictly convex cost function C*

1) *For every profile of AD preferences, the serial demand game is strictly dominance solvable, its Nash equilibrium is strong, and a coalitional Stackelberg equilibrium; the corresponding revelation game is GSP*

2) *The serial demand game $x \rightarrow y$ is the only Anonymous, Smooth, Strictly Monotonic ($\partial_i y_i > 0$) demand game with a unique Nash equilibrium at all profile of AD preferences*

(the full AD domain is necessary for statement 2)

compare with the Average Cost demand game:

$$y_i = \frac{x_i}{x_N} C(x_N)$$

existence of a Nash equilibrium is guaranteed with AD preferences

but uniqueness is only guaranteed if preferences are *binormal* (e.g., quasi-linear)

even then:

- the direct revelation mechanism is manipulable
- the demand game is not dominance solvable, its Nash equilibrium is not strong

normative properties of the SER and AVC equilibria ([22], [21])

→ the serial cost shares meet the lower Stand Alone core and the Unanimity Upper Bound

$$C(x_S) \leq y_S \text{ for all } S \subset N; y_i \leq \frac{1}{n}C(nx_i) \text{ for all } i \in N$$

→ the serial Nash outcome (x_i^*, y_i^*) meets the Unanimity Lower bound

$$u_i(x_i^*) - y_i^* \geq \max_{z_i \geq 0} \{u_i(x_i) - \frac{1}{n}C(nx_i)\}$$

→ it is in the upper SACore: for all $S \subseteq N$

$$\sum_S \{u_i(x_i^*) - y_i^*\} \leq \max \{ \sum_S u_i(x_i) - C(\sum_S x_i) \}$$

→ it is Non Envious: $u_i(x_i^*) - y_i^* \geq u_i(x_j^*) - y_j^*$ for all i, j

SACore and NE compatible for inefficient outcomes!

compare with the Average Cost equilibrium outcome (s):

→ it is in the upper SA core

→ but fails the Unanimity Lower bound

→ and generates Envy

comparing the efficiency loss in the serial (SER) and average cost (AVC) demand games:

- the SER equilibrium Pareto dominates the AVC one at a unanimous utility profile
- the AVC equilibrium cannot Pareto dominate the SER one
- net efficiency losses in equilibrium are not comparable

Price of Anarchy ([19])

worst ratio $\gamma(n, C, \varphi)$ of equilibrium surplus to efficient surplus

→ minimum over all profiles of concave utilities

- for $n = 2$ and $C_p(z) = z^{p+1}$, $\gamma(2, C_p, SER)$ decreases in p from 0.82 to 0.5, while $\gamma(2, C_p, AVC)$ increases from 0.77 to 0.83; crossing at $p = 0.36$

- SER has a much better PoA when n grows large

$$\text{for any } p > 0: \quad \gamma(n, C_p, SER) = \theta\left(\frac{1}{\ln\{n\}}\right); \quad \gamma(n, C_p, AVC) = \theta\left(\frac{1}{n}\right)$$

conjecture: $\theta(\frac{1}{\ln\{n\}})$ is the best asymptotic PoA of any demand game:
budget-balanced division of costs with non negative shares

note: for cost sharing rules allowing *negative* cost shares, an efficient and almost budget balanced method can be constructed, provided the cost function is regular enough (analytic): see [20]

2.3 general supermodular costs

demands $x_i \in \mathbb{N}, \mathbb{R}_+$

concave utility $u_i(x_i) - y_i$

cost function: $C(x_i; i \in N) = \sum_N y_i$

$C(0) = 0$, C is increasing and $\frac{\partial^2 C}{\partial x_i \partial x_j} \geq 0$

discrete model

incremental mechanisms (deterministic)

fix a sequence $\mathbb{N} \ni t \rightarrow i(t) \in N$ such that $|\{t | i(t) = j\}| = \infty$ for all j

offer units at successive marginal costs, in the order of the sequence

an agent is out after first refusal

\Rightarrow *the Theorem still applies*

continuous model

fix a path $\zeta : \mathbb{R}_+ \ni t \rightarrow \zeta(t) \in \mathbb{R}_+^N$, weakly increasing and differentiable,
 $\zeta(\infty) = \infty$

the corresponding cost sharing mechanism:

$$y_i = \int_0^{x_i} \frac{\partial C}{\partial x_i}(\zeta(t) \wedge x) d\zeta(t)$$

The strategic properties of the serial demand and revelation games (statement 1) are preserved

The characterization result still awaits a generalization

3 Production games: submodular costs

3.1 binary heterogeneous demands

each agent demands 0 or 1 unit of service

$N \supseteq S \rightarrow c(S)$ SA cost of serving S

TU game (N, c) is submodular

→ Population Monotonic (aka *Cross Monotonic*) cost sharing rules

$$\varphi_i(S, c) \geq \varphi_i(S \cup \{j\}, c) \text{ for all } i \in S \subset N$$

examples: Shapley value, Dutta-Ray egalitarian core selection

Theorem ([17]) *fix (N, c) submodular*

1) *the demand game has a Pareto dominant strong equilibrium; the associated revelation mechanism is GSP*

2) *these capture all GSP mechanisms meeting*

No Charge for No Demand

Consumer Sovereignty

Theorem ([23]) *Among all above mechanisms, the Shapley value has the smallest worst absolute efficiency loss*

$$\delta = \left\{ \sum_{1 \leq s \leq n} \frac{(s-1)!(n-s)!}{n!} \sum_{S \subseteq N, |S|=s} c(S) \right\} - c(N)$$

example: $c(S) = F + \sum_{i \in S} c_i \Rightarrow$ worst loss $\left\{ \sum_{k=1}^n \frac{F}{k} \right\} - F \simeq F \ln\{n\}$

3.2 multi-units heterogenous demands

sharp contrast binary demands \leftrightarrow multi-units demands

unlike the supermodular case

\rightarrow *existence* of a Nash equilibrium of the demand game is no longer guaranteed on the full AD domain

discrete model (without loss)

agent i demands $x_i \in \mathbb{N}_+$

utility $u_i(x_i)$ is concave

cost function C is submodular: $\frac{\partial^2 C}{\partial x_i \partial x_j} \leq 0$

Cross Monotonic (CM) cost sharing rule φ :

$$\varphi_i(x_i, x_{N \setminus i}; c) \geq \varphi_i(x_i, x'_{N \setminus i}; c) \text{ for all } x_i, x_{N \setminus i} \leq x'_{N \setminus i}$$

or simply $\frac{\partial \varphi_i}{\partial x_j} \leq 0$: my cost share decreases in other agents' demands

Complementarity (COMP) of the rule φ : $\frac{\partial^2 \varphi_i}{\partial x_i \partial x_j} \leq 0$

my cost reduction in other agents' demands decreases in my own demand

examples of cross monotonic sharing rules meeting complementarity

→ incremental demand games:

fix a sequence $\mathbb{N} \ni t \rightarrow i(t) \in N$ such that $|\{t | i(t) = j\}| = \infty$ for all j ;
given demand profile x , charge units at successive costs c_1, c_2, \dots , in the
order of the sequence

→ Shapley Shubik demand games: $\varphi_i(x; c) = E_S\{C(x_S + x_i) - C(x_S)\}$

Lemma *if the rule φ meets CM and COMP, the best reply functions are increasing, so the demand game has a Pareto dominant Nash equilibrium, implemented by the canonical descending algorithm*

→ but this equilibrium **does not** yield a strategyproof revelation mechanism, or a strong equilibrium

example

$n = 2, Q_i = 3$, increments 1, 2, 1, 2, 1, 2

cost $C(x_1 + x_2)$ with $(c_1, \dots, c_6) = (10, 9, 6, 5, 3, 0)$

Ann's marginal utilities: 11, 8, 2; Bob's marginal utilities: 8, 4, 3

descending algorithm: Ann: $x_A = 1 \rightarrow$ Bob: $x_B = 0 \rightarrow$ utilities: $u_A = 1, u_B = 0$

if Ann pretends $x'_A = 3 \rightarrow$ Bob: $x'_B = 3 \rightarrow$ utilities: $u_A = 2, u_B = 1$

3.3 multi-units homogenous demands, dmc costs

continuous model

agent i demands $x_i \in \mathbb{R}_+$

quasi-linear utility $u_i(x_i) - y_i$ concave

$$C(x) = C(\sum_N x_i)$$

$C(0) = 0$, C is increasing and concave

Theorem ([16])

- 1) *the AC demand game has a Nash equilibrium for C such that $C' - AC$ increases, but may not otherwise*
- 2) *SER and SS (Shapley Shubik) have a Pareto dominant Nash equilibrium implemented by the descending algorithm*

→ statement 2) holds for on a larger domain than quasi-linear: binormal preferences

but on the full AD domain, both SER and SS may fail to have a Nash equilibrium

conjecture: on the full Arrow Debreu domain, no demand game guarantees existence of a Nash equilibrium

normative properties of the SER, SS, and AVC equilibria

the equilibrium(a) of each rule, AC, SER, or SS, meets the SA test

the equilibrium of SER

is Non Envious

meets the Unanimity Upper Bound

$$u_i(x_i^*) - y_i^* \leq \max_{z_i \geq 0} \{u_i(x_i) - \frac{1}{n}C(nx_i)\}$$

Theorem ([16]) *the serial rule is the only cross monotonic simple demand game of which all Nash equilibria are Non Envious*

4 General production games

in many important cost sharing problems the cost is merely subadditive, and the upper-SACore may be empty (set cover, vertex cover, traveling salesman, see [27])

a fortiori there is no Cross Monotonic sharing of the cost

fixed priority mechanisms are WGSP and budget balanced, but very unfair, and (very) badly inefficient: we often lose the entire surplus

Random Priority is fair, still SP, but equally inefficient

characterizing all (W)GSP and budget-balance mechanisms is hard, and existing results hard to read: [10]

new idea

combine strategy-proofness with

budget-balance \leftrightarrow allocative efficiency

mechanisms design literature requires 1 out of 2, ignores the other

→ $\text{AEFF} \cap \text{SP}$: the VCG mechanisms

→ $\text{BB} \cap \text{SP}$: the above results

alternative route ([26]): an approximate version of BB and AEFF, and exact SP or (W)GSP

binary demands case

α -budget-balance with a budget deficit

$$\alpha c(S) \leq \sum_S y_i \leq c(S)$$

equivalent results for the budget surplus case

$$c(S) \leq \sum_S y_i \leq \alpha c(S)$$

question: using cross monotonic (\Rightarrow GSP) mechanisms, what BB performance can we guarantee?

example 1: *edge cover problem*

agents are vertices of a connected graph

coalition S is served by any set \mathcal{F} of edges such that every vertex in S is an endpoint of some edge in \mathcal{F}

$$C(\mathcal{F}) = |\mathcal{F}|$$

Proposition ([9]) the best bound is $\alpha = \frac{1}{2}$

example 2: *set cover problem*

edge cover \subset set cover \subset public items problem

N agents; $a \in A \subseteq 2^N$; $c_a = 1$ for all a

$$\mathcal{A}_i = \{B \subset A \mid i \in \cup_B a\}$$

$$c(S) = \min\{|\mathcal{B}| \mid \cup_{B \in \mathcal{B}} a \supseteq S\}$$

Proposition ([9]) an upper bound is $\alpha \leq \frac{K}{n}$

other results include vertex cover, facility location ([25])

standard measure of efficiency performance:

ratio of equilibrium to efficient surplus

example *binary demands case*

$$\frac{\sum_{S^{eq}} u_i - c(S^{eq})}{\sum_{S^{eff}} u_i - c(S^{eff})}$$

this fails because of knife-edge no-surplus cases

use instead the ratio of equilibrium to efficient social cost ([26])

$$\frac{c(S^{eq}) + \sum_{N \setminus S^{eq}} u_i}{c(S^{eff}) + \sum_{N \setminus S^{eff}} u_i}$$

acyclic mechanisms ([13]) generalize cross monotonic ones by offering cost shares in turn, and updating offers as soon as anyone drops

include fixed priority mechanisms, and much more

→ ensure WGSP

→ better α and β performance

example set cover

$$\text{CM mechas} \Rightarrow \alpha \leq \frac{K}{n}, \beta \leq \frac{K'}{\sqrt{n}}$$

$$\text{Acyclic mechas} \Rightarrow \alpha, \beta \geq \frac{K}{\ln\{n\}}$$

→ extend to multi-units demands

example: symmetric technology with U-shaped average cost, 3 agents

$$c_1 = 10, c_2 = 12, c_3 = 24$$

$$u_1 = 9, \quad = u_3 = 7; u_2 = 5$$

efficiency: $S^{eff} = \{1, 3\}$, $v(N) = 4$

Cross Monotonic mechanism

offer $\frac{c_3}{3} = 8$ to all \rightarrow 2, 3 decline \rightarrow offer c_1 to 1 who declines \rightarrow zero surplus

Acyclic mechanism

offer $\frac{c_3}{3}$ to 1 \rightarrow accepts \rightarrow offer $\frac{c_3}{3}$ to 2 \rightarrow declines \rightarrow offer $\frac{c_2}{2}$ to 1 \rightarrow
accepts \rightarrow offer $\frac{c_2}{2}$ to 3 \rightarrow accepts \rightarrow efficient surplus

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