FAIR DIVISION 4: INDIVISIBLE GOODS
Indivisible goods

- Set $G$ of $m$ goods
- Each good is indivisible
- Players $N = \{1, \ldots, n\}$ have arbitrary valuations $V_i$ for bundles of goods
- Envy-freeness and proportionality are infeasible!
**Minimizing envy**

- Given allocation $A$, denote
  
  $$e_{ij}(A) = \max\{0, V_i(A_j) - V_i(A_i)\}$$
  
  $$e(A) = \max\{e_{ij}(A): i, j \in N\}$$

- Theorem [Nisan and Segal 2002]: Every protocol that finds an allocation minimizing $e(A)$ must use an exponential number of bits of communication in the worst case.
**Communication complexity**

- Protocol defined by a binary tree
- Complexity is the height of the tree
- Complexity of a problem is the height of the shortest tree
Proof of theorem

• Let \( m = 2k \)

• \( \mathcal{F} \) is a set of functions s.t. for all \( V \in \mathcal{F}, \ S \subseteq G, \)

\[
V(S) = \begin{cases} 
1 & |S| > k \\
0 & |S| < k \\
1 - V(G \setminus S) & |S| = k 
\end{cases}
\]

• \( |\mathcal{F}| = 2^{\binom{m}{k}} \)

Proof of theorem

• Suppose $n = 2$, and denote a valuation profile by $(U, V) \in \mathcal{F}^2$

• Lemma: Suppose $U \in \mathcal{F}, V \in \mathcal{F} \setminus \{U\}$, then the sequence of bits transmitted on input $(U, U)$ is different from the sequence transmitted on $(V, V)$

• Assume the lemma is true, then there must be at least $|\mathcal{F}|$ sequences, and the height of the tree must be at least $\log |\mathcal{F}| = \binom{m}{k}/2$
Proof of lemma

- Assume not; then \((U, V)\) and \((V, U)\) generate the same sequence
\[(U, U)\]

\[(V, V)\]

\[(U, V)\]
Proof of lemma

• If $U \neq V$, $\exists T \subset G$ such that $U(T) = 1$, $V(T) = 0$
• The allocation $(T, G \setminus T)$ is EF for $(U, V)$, $(G \setminus T, T)$ is EF for $(V, U)$
• Given $(U, V)$, protocol produces an EF $(S, G \setminus S) \Rightarrow U(S) = 1$, $V(G \setminus S) = 1$
• $(S, G \setminus S)$ is also returned on $(V, U)$, but is not EF
Approximate EF

- Define the maximum marginal utility
  \[ \alpha = \max \{ V_i(S \cup \{x\}) - V_i(S) : i, x, S \} \]
- Theorem [Lipton et al. 2004]: An allocation with \( e(A) \leq \alpha \) can be found in polynomial time
- Note: we are still not assuming anything about the valuation functions!
Proof of Theorem

• Given allocation $A$, we have an edge $(i, j)$ in its envy graph if $i$ envies $j$

• Lemma: Given partial allocation $A$ with envy graph $G$, can find allocation $B$ with acyclic envy graph $H$ s.t. $e(B) \leq e(A)$
Proof of lemma

• If $G$ has a cycle $C$, shift allocations along $C$ to obtain $A'$; clearly $e(A') \leq e(A)$

• #edges in envy graph of $A'$ decreased:
  o Same edges between $N \setminus C$
  o Edges from $N \setminus C$ to $C$ shifted
  o Edges from $C$ to $N \setminus C$ can only decrease
  o Edges inside $C$ decreased

• Iteratively remove cycles ■
Proof of theorem

• Maintain envy $\leq \alpha$ and acyclic graph
• In round 1, allocate good $g_1$ to arbitrary agent
• $g_1, \ldots, g_{k-1}$ are allocated in acyclic $A$
• Derive $B$ by allocating $g_k$ to source $i$
• $e_{ji}(B) \leq e_{ji}(A) + \alpha = \alpha$
• Use lemma to eliminate cycles ■
EF CAKE CUTTING, REVISITED

• Want to get $\epsilon$-EF cake division

• Agent $i$ makes $1/\epsilon$ marks $x_1^i, \ldots, x_{1/\epsilon}^i$ such that for every $k$, $V_i([x_k^i, x_{k+1}^i]) = \epsilon$

• If intervals between consecutive marks are indivisible goods then $\alpha \leq \epsilon$

• Now we can apply the theorem

• Need $n/\epsilon$ cut queries and $n^2/\epsilon$ eval queries
An even simpler solution

• Relies on **additive** valuations
• Create the “indivisible goods” like before
• Agents choose pieces in a round-robin fashion: $1, \ldots, n, 1, \ldots, n, \ldots$
• Each good chosen by agent $i$ is preferred to the next good chosen by agent $j$
• This may not account for the first good $g$ chosen by $j$, but $V_i(\{g\}) \leq \epsilon$
Maximin share guarantee

• Let us focus on indivisible goods and additive valuations
• Communication complexity is not an issue
• But computational complexity is
• Observation: Deciding whether there exists an EF allocation is NP-hard, even for two players with identical additive valuations
### Maximin Share Guarantee

| Total:  
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$30</td>
<td>$50</td>
<td>$20</td>
<td>$30</td>
<td>$30</td>
<td>$50</td>
<td>$20</td>
<td>$30</td>
</tr>
<tr>
<td>$30</td>
<td>$50</td>
<td>$5</td>
<td>$5</td>
<td>$3</td>
<td>$5</td>
<td>$3</td>
<td>$5</td>
</tr>
</tbody>
</table>

Total:
- $30
- $50
- $20
- $30
- $30
- $50
- $20
- $30

15896 Spring 2016: Lecture 9
• Maximin share (MMS) guarantee [Budish, 2011] of player $i$:
  \[ \max_{x_1, \ldots, x_n} \min_j V_i(X_j) \]

• Theorem [P & Wang, 2014]: $\forall n \geq 3$ there exist additive valuation functions that do not admit an MMS allocation
**Counterexample for** $n = 3$

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>17</td>
<td>25</td>
<td>12</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>22</td>
<td>3</td>
<td>28</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>0</td>
<td>21</td>
<td>23</td>
<td></td>
</tr>
</tbody>
</table>
Counterexample for $n = 3$

$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \times 10^6 + \begin{bmatrix} 17 & 25 & 12 & 1 \\ 2 & 22 & 3 & 28 \\ 11 & 0 & 21 & 23 \end{bmatrix} \times 10^3 + \begin{bmatrix} 3 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Player 1

$\begin{bmatrix} 3 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Player 2

$\begin{bmatrix} 3 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

Player 3
• Maximin share (MMS) guarantee [Budish, 2011] of player $i$:

$$\max_{x_1,\ldots,x_n} \min_j V_i(X_j)$$

• Theorem [P & Wang, 2014]: \(\forall n \geq 3\) there exist additive valuation functions that do not admit an MMS allocation

• Theorem [P & Wang, 2014]: It is always possible to guarantee each player \(2/3\) of his MMS guarantee