CMU 15-896
Noncooperative games 3: Price of anarchy

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Back to prison

- The only Nash equilibrium in Prisoner’s dilemma is bad; but how bad is it?
- Objective function: social cost = sum of costs
- NE is six times worse than the optimum

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<th>Cooperate</th>
<th>Defect</th>
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<tr>
<td>Defect</td>
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<td>-6,-6</td>
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Anarchy and Stability

- Fix a class of games, an objective function, and an equilibrium concept
- The price of anarchy (stability) is the worst-case ratio between the worst (best) objective function value of an equilibrium of the game, and that of the optimal solution
- In this lecture:
  - Objective function = social cost
  - Equilibrium concept = Nash equilibrium
**Example: Cost sharing**

- \( n \) players in weighted directed graph \( G \)
- Player \( i \) wants to get from \( s_i \) to \( t_i \); strategy space is \( s_i \rightarrow t_i \) paths
- Each edge \( e \) has cost \( c_e \)
- Cost of edge is split between all players using edge
- Cost of player is sum of costs over edges on path
Example: Cost sharing

- With $n$ players, the example on the right has an NE with social cost $n$
- Optimal social cost is 1
- $\Rightarrow$ Price of anarchy $\geq n$

Prove that the price of anarchy is at most $n$
Example: Cost sharing

- Think of the 1 edges as cars, and the $k$ edge as mass transit
- Bad Nash equilibrium with cost $n$
- Good Nash equilibrium with cost $k$
- Now let’s modify the example...
Example: Cost sharing

- OPT = $k + 1$
- Only equilibrium has cost $k \cdot H(n)$
- $\Rightarrow$ price of stability is at least $\Omega(\log n)$
- We will show that the price of stability is $\Theta(\log n)$
Potential games

A game is an exact potential game if there exists a function $\Phi: \prod_{i=1}^{n} S_i \rightarrow \mathbb{R}$ such that for all $i \in N$, for all $s \in \prod_{i=1}^{n} S_i$, and for all $s_i' \in S_i$,

$$\text{cost}_i(s_i', s_{-i}) - \text{cost}_i(s) = \Phi(s_i', s_{-i}) - \Phi(s)$$

Why does the existence of an exact potential function imply the existence of a pure Nash equilibrium?
Potential games

- Theorem: the cost sharing game is an exact potential game

- Proof:
  - Let $n_e(s)$ be the number of players using $e$ under $s$
  - Define the potential function
    $$\Phi(s) = \sum_e \sum_{k=1}^{n_e(s)} \frac{c_e}{k}$$
  - If player changes paths, pays $\frac{c_e}{n_e(s)+1}$ for each new edge, gets $\frac{c_e}{n_e(s)}$ for each old edge, so $\Delta \text{cost}_i = \Delta \Phi$ ■
Potential games

• Theorem: The cost of stability of cost sharing games is $O(\log n)$

• Proof:
  o It holds that
    \[ \text{cost}(s) \leq \Phi(s) \leq H(n) \cdot \text{cost}(s) \]
  o Take a strategy profile $s$ that minimizes $\Phi$
  o $s$ is an NE
  o $\text{cost}(s) \leq \Phi(s) \leq \Phi(\text{OPT}) \leq H(n) \cdot \text{cost(OPT)}$
Cost sharing summary

• In every cost sharing game
  o $\forall$NE $s$, $\text{cost}(s) \leq n \cdot \text{cost(OPT)}$
  o $\exists$NE $s$ such that $\text{cost}(s) \leq H(n) \cdot \text{cost(OPT)}$

• There exist cost sharing games s.t.
  o $\exists$NE $s$ such that $\text{cost}(s) \geq n \cdot \text{cost(OPT)}$
  o $\forall$NE $s$, $\text{cost}(s) \geq H(n) \cdot \text{cost(OPT)}$
Congestion games

• Generalization of cost sharing games
• $n$ players and $m$ resources
• Each player $i$ chooses a set of resources (e.g., a path) from collection $S_i$ of allowable sets of resources (e.g., paths from $s_i$ to $t_i$)
• Cost of resource $j$ is a function $f_j(n_j)$ of the number $n_j$ of players using it
• Cost of player is the sum over used resources
Congestion games

- Theorem [Rosenthal 1973]: Every congestion game is an exact potential game

- Proof: The exact potential function is

  \[ \Phi(s) = \sum_j \sum_{i=1}^{n_j(s)} f_j(i) \]

- Theorem [Monderer and Shapley 1996]: Every potential game is isomorphic to a congestion game
Network formation games

• Each player is a vertex \( v \)
• Strategy of \( v \): set of undirected edges to build that touch \( v \)
• Strategy profile \( s \) induces undirected graph \( G(s) \)
• Cost of building any edge is \( \alpha \)
• \( \text{cost}_v(s) = \alpha n_v(s) + \sum_u d(u, v) \), where \( n_v = \) #edges bought by \( v \), \( d \) is shortest path in #edges
• \( \text{cost}(s) = \sum_{u \neq v} d(u, v) + \alpha |E| \)
Example: Network formation

• NE with $\alpha = 3$

Suboptimal

Optimal
Example: Network formation

• **Lemma:** If \( \alpha \geq 2 \) then any star is optimal, and if \( \alpha \leq 2 \) then a complete graph is optimal

• **Proof:**
  - Suppose \( \alpha \leq 2 \), and consider any graph that is not complete
  - Adding an edge will decrease the sum of distances by at least 2, and costs only \( \alpha \)
  - Suppose \( \alpha \geq 2 \) and the graph contains a star, so the diameter is at most 2; deleting a non-star edge increases the sum of distances by at most 2, and saves \( \alpha \)

\[\square\]
Example: Network formation

Poll: For which values of $\alpha$ is any star a NE, and for which is the complete graph a NE?

1. $\alpha \geq 1, \alpha \leq 1$
2. $\alpha \geq 2, \alpha \leq 1$
3. $\alpha \geq 1$, none
4. $\alpha \geq 2$, none
Example: Network formation

- Theorem:
  1. If $\alpha \geq 2$ or $\alpha \leq 1$, PoS = 1
  2. For $1 < \alpha < 2$, PoS $\leq 4/3$

- Proof:
  - Part 1 is immediate from the lemma and poll
  - For $1 < \alpha < 2$, the star is a NE, while OPT is a complete graph
  - Worst case ratio when $\alpha \to 1$:
    \[
    \frac{2n(n - 1) - (n - 1)}{n(n - 1) + n(n - 1)/2} = \frac{4n^2 - 6n + 2}{3n^2 - 3n} < \frac{4}{3}
    \]
Example: Network creation

- Theorem [Fabrikant et al. 2003]: The price of anarchy of network creation games is $O(\sqrt{\alpha})$

- Lemma: If $s$ is a Nash equilibrium that induces a graph of diameter $d$, then $\text{cost}(s) \leq O(d) \cdot \text{OPT}$
Proof of lemma

• $\text{OPT} = \Omega(\alpha n + n^2)$
  
  o Buying a connected graph costs at least $(n - 1)\alpha$
  
  o There are $\Omega(n^2)$ distances

• Distance costs $\leq dn^2 \Rightarrow$ focus on edge costs

• There are at most $n - 1$ cut edges $\Rightarrow$ focus on noncut edges
**Proof of lemma**

- **Claim:** Let $e = (u, v)$ be a noncut edge, then the distance $d(u, v)$ with $e$ deleted $\leq 2d$
  - $V_e =$ set of nodes s.t. the shortest path from $u$ uses $e$
  - Figure shows shortest path avoiding $e$, $e' = (u', v')$ is the edge on the path entering $V_e$
  - $P_u$ is the shortest path from $u$ to $u' \Rightarrow |P_u| \leq d$
  - $|P_v| \leq d - 1$ as $P_v \cup e$ is shortest path from $u$ to $v'$
Proof of lemma

• Claim: There are $O(nd/\alpha)$ noncut edges paid for by any vertex $u$
  o Let $e = (u, v)$ be an edge paid for by $u$
  o By previous claim, deleting $e$ increases distances from $u$ by at most $2d|V_e|$.
  o $G$ is an equilibrium $\Rightarrow \alpha \leq 2d|V_e| \Rightarrow |V_e| \geq \alpha / 2d$
  o $n$ vertices overall $\Rightarrow$ can’t be more than $2nd/\alpha$
    sets $V_e$ ■
Proof of lemma

- $O(nd/\alpha)$ noncut edges per vertex
- $O(nd)$ total payment for these per vertex
- $O(n^2d)$ overall
Proof of theorem

• By lemma, it is enough to show that the diameter at a NE $\leq 2\sqrt{\alpha}$
• Suppose $d(u, v) \geq 2k$ for some $k$
• By adding the edge $(u, v)$, $u$ pays $\alpha$ and improves distance to second half of the $u \rightarrow v$ shortest path by
  $$(2k - 1) + (2k - 3) + \cdots + 1 = k^2$$
• If $d(u, v) > 2\sqrt{\alpha}$, it is beneficial to add edge