CMU 15-896
Noncooperative games 1: Basic concepts

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Normal-Form Game

- A game in normal form consists of:
  - Set of players $N = \{1, ..., n\}$
  - Strategy set $S$
  - For each $i \in N$, utility function $u_i : S^n \rightarrow \mathbb{R}$: if each $j \in N$ plays the strategy $s_j \in S$, the utility of player $i$ is $u_i(s_1, ..., s_n)$

- Next example created by taking screenshots of http://youtu.be/jILgxeNBK_8
Selling ice cream at the beach.

One day your cousin Ted shows up.

His ice cream is identical!

You split the beach in half; you set up at 1/4.

50% of the customers buy from you.

50% buy from Ted.

One day Ted sets up on the 1/2 point!

Now you serve only 37.5%!
The Ice Cream Wars

- $N = \{1,2\}$
- $S = [0,1]$  
  \[
  u_i(s_i, s_j) = \begin{cases} 
    \frac{s_i + s_j}{2}, & s_i < s_j \\
    1 - \frac{s_i + s_j}{2}, & s_i > s_j \\
    \frac{1}{2}, & s_i = s_j
  \end{cases}
  
- To be continued...
The prisoner’s dilemma

- Two men are charged with a crime
- They are told that:
  - If one rats out and the other does not, the rat will be freed, other jailed for nine years
  - If both rat out, both will be jailed for six years
- They also know that if neither rats out, both will be jailed for one year
### The prisoner’s dilemma

<table>
<thead>
<tr>
<th></th>
<th>Cooperate</th>
<th>Defect</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cooperate</td>
<td>-1, -1</td>
<td>-9, 0</td>
</tr>
<tr>
<td>Defect</td>
<td>0, -9</td>
<td>-6, -6</td>
</tr>
</tbody>
</table>

What would you do?
Prisoner’s dilemma on TV

http://youtu.be/S0qjK3TWZE8
## The Professor’s Dilemma

A professor must decide whether to make an effort or slack off. The professor’s choices are:

- **Make effort**
- **Slack off**

The professor is evaluated based on whether they attend class or not:

- **Attend class**: 10^6, 10^6 (positive outcome for both professor and class)
- **Do not attend class**: -10, 0 (negative outcome for the professor)

### Table

<table>
<thead>
<tr>
<th>Professor Action</th>
<th>Class Attendance</th>
<th>Outcome</th>
</tr>
</thead>
<tbody>
<tr>
<td>Make effort</td>
<td>Listen</td>
<td>10^6, 10^6</td>
</tr>
<tr>
<td></td>
<td>Sleep</td>
<td>-10, 0</td>
</tr>
<tr>
<td>Slack off</td>
<td>Listen</td>
<td>0, -10</td>
</tr>
<tr>
<td></td>
<td>Sleep</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

### Dominant Strategies?

To determine if there are dominant strategies, compare the outcomes for each action:

- **Make effort**
  - Attend class: 10^6, 10^6
  - Do not attend class: -10, 0
- **Slack off**
  - Attend class: 0, -10
  - Do not attend class: 0, 0

There are no dominant strategies.
Nash equilibrium

• Each player’s strategy is a best response to strategies of others

• Formally, a Nash equilibrium is a vector of strategies $s = (s_1, \ldots, s_n) \in S^n$ such that
  $$\forall i \in N, \forall s_i' \in S, u_i(s) \geq u_i(s_i', s_{-i})$$
Nash equilibrium

http://youtu.be/CemLiSI5ox8
Russel Crowe was wrong

Hey, Dr. Nash, I think those gals over there are eyeing us. This is like your Nash equilibrium, right? One of them is hot, but we should each flirt with one of her less-desirable friends. Otherwise we risk coming on too strong to the hot one and just driving the group off.

Well, that's not really the sort of situation I wrote about. Once we're with the ugly ones, there's no incentive for one of us not to try to switch to the hot one. It's not a stable equilibrium.

Crap, forget it. Looks like all three are leaving with one guy. Dammit, Feynman!
## Rock-paper-scissors

<table>
<thead>
<tr>
<th></th>
<th>R</th>
<th>P</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td>0,0</td>
<td>-1,1</td>
<td>1,-1</td>
</tr>
<tr>
<td>P</td>
<td>1,-1</td>
<td>0,0</td>
<td>-1,1</td>
</tr>
<tr>
<td>S</td>
<td>-1,1</td>
<td>1,-1</td>
<td>0,0</td>
</tr>
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</table>

Nash equilibrium?
**Mixed strategies**

- A **mixed strategy** is a probability distribution over (pure) strategies.
- The mixed strategy of player $i \in N$ is $x_i$, where
  
  $$x_i(s_i) = \Pr[i \text{ plays } s_i]$$

- The utility of player $i \in N$ is
  
  $$u_i(x_1, \ldots, x_n) = \sum_{(s_1, \ldots, s_n) \in S^n} u_i(s_1, \ldots, s_n) \cdot \prod_{j=1}^{n} x_j(s_j)$$
Nash’s Theorem

• Theorem [Nash, 1950]: if everything is finite then there exists at least one (possibly mixed) Nash equilibrium

• We’ll talk about computation some other time
Does NE make sense?

- Two players, strategies are \( \{2, \ldots, 100\} \)
- If both choose the same number, that is what they get
- If one chooses \( s \), the other \( t \), and \( s < t \), the former player gets \( s + 2 \), and the latter gets \( s - 2 \)
- Poll 1: what would you choose?
**Correlated equilibrium**

- Let $N = \{1,2\}$ for simplicity
- A mediator chooses a pair of strategies $(s_1, s_2)$ according to a distribution $\rho$ over $S^2$
- Reveals $s_1$ to player 1 and $s_2$ to player 2
- When player 1 gets $s_1 \in S$, he knows that the distribution over strategies of 2 is

$$
\Pr[s_2|s_1] = \frac{\Pr[s_1 \land s_2]}{\Pr[s_1]} = \frac{\rho(s_1, s_2)}{\sum_{s_2' \in S} \rho(s_1, s_2')}
$$
Correlated equilibrium

• Player 1 is best responding if for all $s'_1 \in S$
  $$\sum_{s_2 \in S} \Pr[s_2|s_1]u_1(s_1, s_2) \geq \sum_{s_2 \in S} \Pr[s_2|s_1]u_1(s'_1, s_2)$$

• Equivalently,
  $$\sum_{s_2 \in S} p(s_1, s_2)u_1(s_1, s_2) \geq \sum_{s_2 \in S} p(s_1, s_2)u_1(s'_1, s_2)$$

• $p$ is a correlated equilibrium (CE) if both players are best responding
Game of chicken

http://youtu.be/u7hZ9jKrwvo
Game of chicken

- **Social welfare** is the sum of utilities
- Pure NE: (C,D) and (D,C), social welfare = 5
- Mixed NE: both (1/2, 1/2), social welfare = 4
- Optimal social welfare = 6

<table>
<thead>
<tr>
<th></th>
<th>Dare</th>
<th>Chicken</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dare</td>
<td>0,0</td>
<td>4,1</td>
</tr>
<tr>
<td>Chicken</td>
<td>1,4</td>
<td>3,3</td>
</tr>
</tbody>
</table>
Game of chicken

- Correlated equilibrium:
  - (D,D): 0
  - (D,C): $\frac{1}{3}$
  - (C,D): $\frac{1}{3}$
  - (C,C): $\frac{1}{3}$

- Social welfare of CE $= \frac{16}{3}$
Implementation of CE

- Instead of a mediator, use a hat!
- Balls in hat are labeled with “chicken” or “dare”, each blindfolded player takes a ball

Which balls implement the distribution of the previous slide?
CE vs. NE

• Poll 2: What is the relation between CE and NE?

1. CE $\Rightarrow$ NE
2. NE $\Rightarrow$ CE
3. NE $\Leftrightarrow$ CE
4. NE $\parallel$ CE
CE As LP

• Can compute CE via linear programming in polynomial time!

\[
\text{find } \forall s_1, s_2 \in S, p(s_1, s_2) \\
\text{s.t. } \forall s_1, s_1', s_2 \in S, \sum_{s_2 \in A} p(s_1, s_2)u_1(s_1, s_2) \geq \sum_{s_2 \in A} p(s_1, s_2)u_1(s_1', s_2) \\
\forall s_1, s_2, s_2' \in S, \sum_{s_1 \in A} p(s_1, s_2)u_2(s_1, s_2) \geq \sum_{s_1 \in A} p(s_1, s_2)u_2(s_1, s_2') \\
\sum_{s_1, s_2 \in S} p(s_1, s_2) = 1 \\
\forall s_1, s_2 \in S, p(s_1, s_2) \in [0,1]
\]