Kidney Exchange

Donor 1 → Patient 2
Donor 2 → Patient 1

Donor 1
Patient 1

Donor 2
Patient 2
**Incentives**

- A decade ago kidney exchanges were carried out by individual hospitals.
- Today there are nationally organized exchanges; participating hospitals have little other interaction.
- It was observed that hospitals match easy-to-match pairs internally, and enroll only hard-to-match pairs into larger exchanges.
- Goal: incentivize hospitals to enroll all their pairs.
The strategic model

• Undirected graph (only pairwise matches!)
  o Vertices = donor-patient pairs
  o Edges = compatibility
  o Each player controls subset of vertices
• Mechanism receives a graph and returns a matching
• Utility of player = # its matched vertices
• Target: # matched vertices
• Strategy: subset of revealed vertices
  o But edges are public knowledge
• Mechanism is strategyproof (SP) if it is a dominant strategy to reveal all vertices
OPT is manipulable
OPT is manipulable
Approximating SW

- Theorem [Ashlagi et al. 2010]: No deterministic SP mechanism can give a $2 - \epsilon$ approximation
- Proof: We just proved it!
- Theorem [Kroer and Kurokawa 2013]: No randomized SP mechanism can give a $\frac{6}{5} - \epsilon$ approximation
- Proof: Homework 2
SP mechanism: Take 1

• Assume two players

• The \text{MATCH}_{\{1\},\{2\}} mechanism:
  
  o Consider matchings that maximize the number of “internal edges”
  
  o Among these return a matching with max cardinality
Another example

\[ \text{Diagram showing connected nodes with checks.} \]
Guarantees

- \text{MATCH}_{\{1\},\{2\}} \text{ gives a 2-approximation}
  - Cannot add more edges to matching
  - For each edge in optimal matching, one of the two vertices is in mechanism’s matching

- Theorem (special case): \text{MATCH}_{\{1\},\{2\}} \text{ is strategyproof for two players}
Proof of theorem

- \( M \) = matching when player 1 is honest, \( M' \) = matching when player 1 hides vertices
- \( M \Delta M' \) consists of paths and even-length cycles, each consisting of alternating \( M, M' \) edges

What’s wrong with the illustration on the right?
Proof of theorem

- Consider a path in $M \Delta M'$, denote its edges in $M$ by $P$ and its edges in $M'$ by $P'$
- For $i, j \in \{1,2\}$,
  \[ P_{ij} = \{ (u, v) \in P : u \in V_i, v \in V_j \} \]
  \[ P'_{ij} = \{ (u, v) \in P' : u \in V_i, v \in V_j \} \]
- $|P_{11}| \geq |P'_{11}|$, suppose $|P_{11}| = |P'_{11}|$
- It holds that $|P_{22}| = |P'_{22}|$
- $M$ is max cardinality $\Rightarrow |P_{12}| \geq |P'_{12}|$
- $U_1(P) = 2|P_{11}| + |P_{12}| \geq 2|P'_{11}| + |P'_{12}| = U_1(P')$
**Proof of theorem**

- Suppose $|P_{11}| > |P'_{11}|$
- $|P_{12}| \geq |P'_{12}| - 2$
  - Every subpath within $V_2$ is of even length
  - We can pair the edges of $P_{12}$ and $P'_{12}$, except maybe the first and the last
- $U_1(P) = 2|P_{11}| + |P_{12}| \geq 2(|P'_{11}| + 1) + |P'_{12}| - 2 = U_1(P')$
The case of 3 players
SP mechanism: Take 2

• Let $\Pi = (\Pi_1, \Pi_2)$ be a bipartition of the players

• The $\text{MATCH}_\Pi$ mechanism:
  
  o Consider matchings that maximize the number of “internal edges” and do not have any edges between different players on the same side of the partition
  
  o Among these return a matching with max cardinality (need tie breaking)
Eureka?

- Theorem [Ashlagi et al. 2010]: MATCH$\Pi$ is strategyproof for any number of players and any partition $\Pi$
- Recall: for $n = 2$, MATCH$\{{1}\},\{{2}\}$ guarantees a 2-approx
Eureka?

Poll 1: approximation guarantees given by $\text{MATCH}_\Pi$ for $n = 3$ and $\Pi = \{\{1\}, \{2,3\}\}$?

1. 2
2. 3
3. 4
4. More than 4
The mechanism

- The MIX-AND-MATCH mechanism:
  - Mix: choose a random partition $\Pi$
  - Match: Execute $\text{MATCH}_\Pi$
- Theorem [Ashlagi et al. 2010]: MIX-AND-MATCH is strategyproof and guarantees a 2-approximation
- We only prove the approximation ratio
Proof of theorem

- $M^* =$ optimal matching
- Create a matching $M'$ such that $M'$ is max cardinality on each $V_i$, and
  \[
  \sum_i |M'_{ii}| + \frac{1}{2} \sum_{i \neq j} |M'_{ij}| \geq \sum_i |M^*_{ii}| + \frac{1}{2} \sum_{i \neq j} |M^*_{ij}|
  \]
  - $M^{**} =$ max cardinality on each $V_i$
  - For each path $P$ in $M^* \Delta M^{**}$, add $P \cap M^{**}$ to $M'$ if $M^{**}$ has more internal edges than $M^*$, otherwise add $P \cap M^*$ to $M'$
  - For every internal edge $M'$ gains relative to $M^*$, it loses at most one edge overall \(\blacksquare\)
Proof of theorem

• Fix $\Pi$ and let $M^\Pi$ be the output of $\text{MATCH}_\Pi$

• The mechanism returns max cardinality across $\Pi$ subject to being max cardinality internally, therefore

$$\sum_i |M^\Pi_{ii}| + \sum_{i \in \Pi_1, j \in \Pi_2} |M^\Pi_{ij}| \geq \sum_i |M'_i| + \sum_{i \in \Pi_1, j \in \Pi_2} |M'_{ij}|$$
Proof of theorem

\[
E[|M^\Pi|] = \frac{1}{2^n} \sum_{\Pi} \left( \sum_i |M_{ii}| + \sum_{i \in \Pi_1, j \in \Pi_2} |M_{ij}| \right)
\geq \frac{1}{2^n} \sum_{\Pi} \left( \sum_i |M'_{ii}| + \sum_{i \in \Pi_1, j \in \Pi_2} |M'_{ij}| \right)
= \sum_i |M'_{ii}| + \frac{1}{2^n} \sum_{\Pi} \sum_{i \in \Pi_1, j \in \Pi_2} |M'_{ij}|
= \sum_i |M'_{ii}| + \frac{1}{2} \sum_{i \neq j} |M'_{ij}| \geq \sum_i |M^*_{ii}| + \frac{1}{2} \sum_{i \neq j} |M^*_{ij}|
\geq \frac{1}{2} \sum_i |M^*_{ii}| + \frac{1}{2} \sum_{i \neq j} |M^*_{ij}| = \frac{1}{2} |M^*| \quad \square
\]