Cooperative Games –
The Shapley value and Weighted Voting

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The Shapley Value

Given a player $i$, and a set $S \subseteq N$, the marginal contribution of $i$ to $S$ is

$$m_i(S) = \nu(S \cup \{i\}) - \nu(S)$$

How much does $i$ contribute by joining $S$?

Given a permutation $\sigma \in \Pi(N)$ of players, let the predecessors of $i$ in $\sigma$ be

$$P_i(\sigma) = \{j \in N \mid \sigma(j) < \sigma(i)\}$$

We write $m_i(\sigma) = m_i(P_i(\sigma))$
The Shapley Value

Suppose that we choose an ordering of the players uniformly at random. The Shapley value of player $i$ is

$$\phi_i = \mathbb{E}[m_i(\sigma)] = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m_i(\sigma)$$

$q = 50$
The Shapley Value

Efficient: $\sum_{i \in N} \phi_i = \nu(N)$

Symmetric: players who contribute the same are paid the same.

Dummy: dummy players aren’t paid.

Additive: $\phi_i(G_1) + \phi_i(G_2) = \phi_i(G_1 + G_2)$

The Shapley value is the only payoff division satisfying all of the above!
The Shapley Value

Theorem: if a value satisfies efficiency, additivity, dummy and symmetry, then it is the Shapley value.

Proof: let’s prove it on the board.
Computing Power Indices

The Shapley value has a brother – the Banzhaf value

\[ \beta_i = \frac{1}{2^n} \sum_{S \subseteq N} m_i(S) \]

It uniquely satisfies a different set of axioms

Different distributional assumption – more biased towards sets of size \( \frac{n}{2} \)
Voting Power in the EU Council of Members

- The EU council of members is one of the governing members of the EU.
  - Each state has a number of representatives proportional to its population
  - Proportionality: “one person – one vote”

- In terms of voting power - $\phi_i \sim \frac{w_i}{w(N)}$
Voting Power in the EU Council of Members

- Changes to the voting system can achieve better proportional representation.

- **Changing the weights** – generally unpopular and politically delicate

- **Changing the quota** – easier to do, an “innocent” change.

  Selecting an appropriate quota (EU - about 62%), achieves proportional representation with a very small error!
Changing the Quota

• Changes to the quota change players’ power.

• What is the relation between quota selection and voting power?

\[ i \Rightarrow i(q) \]
A “typical” graph of $i(q)$

The graph converges to some value when quota is 50%.

Max at $w_i$

Lower variation towards the 50% quota

Min at $w_i + 1$

The graph converges to some value when quota is 50%...
Weights are a Fibonacci Series
Maximizing $\phi_i(q)$

Theorem: $\phi_i(q)$ is maximized at $q = w_i$

**Proof:** two cases

$q \leq w_i$: if $i$ is pivotal for $\sigma \in \Pi(N)$ under $q$ then $w(P_i(\sigma)) < q \leq w_i$, but $w(P_i(\sigma)) + w_i \geq q$. This implies that $i$ is pivotal for $\sigma$ when the threshold is $w_i$ as well.
Maximizing $\phi_i(q)$

**Lemma:** let $T_i(x) = \{\sigma \in \Pi(N) \mid w(P_i(\sigma)) < x\}$

Then

$$|T_i(x)| + |T_i(y)| \geq |T_i(x + y)|$$

for all $x, y \in \mathbb{N}$

**Proof:** assume that $x \geq y$. We write

$$T_i(x, y) = \{\sigma \in \Pi(N) \mid x \leq w(P_i(\sigma)) < y\}$$

$T_i(x) \subseteq T_i(x + y)$, so $T_i(x + y) \setminus T_i(x) = T_i(x, x + y)$

Need to show that $|T_i(y)| \geq |T_i(x, x + y)|$
Maximizing $\phi_i(q)$

Need to show that $|T_i(y)| \geq |T_i(x, x + y)|$

Construct an injective mapping $\psi: T_i(x, x + y) \rightarrow T_i(y)$

$A: w(A) > x$  $B$

$B$  $A: w(A) > x$
Maximizing $\phi_i(q)$

Second case: $q > w_i$
Let $\Pi_i(q) = \{\sigma \in \Pi(N) \mid q - w_i \leq w(P_i(\sigma)) < q\}$, then
$\Pi_i(q) = T_i(q - w_i, q)$ and $\Pi_i(w_i) = T_i(w_i)$.

By Lemma

$$|\Pi_i(w_i)| = |T_i(w_i)| \geq |T_i(q)| - |T_i(q - w_i)| = |\Pi_i(q)|$$

which concludes the proof.
Minimizing $\phi_i(q)$

Not as easy, two strong candidate minimizers: $q = 1$ or $q = w_i + 1$.

Not always them, not clear which one to choose. For below-median players, setting $q = w_i + 1$ is worse.

Deciding whether a given quota is maximizing/minimizing is computationally intractable.
The expected behavior of $\phi_i(q)$

It seems that analyzing fixed weight vectors is not very effective...

even small changes in quota can cause unpredictable behavior; worst-case guarantees are not great.

Can we say something about the likely Shapley value when weights are sampled from a distribution?
Balls and Bins Distributions

• We have $m$ balls, $n$ bins.
• A discrete probability distribution $(p_1, \ldots, p_n)$
• $p_i$ is the probability that a ball will land in bin $i$
Huge disparity at some thresholds

Near Equality at others...

Changing the threshold from 500 to 550 results in a huge shift in voting power
Balls and Bins: Uniform

• Suppose that the weights are generated from a uniform balls and bins process with m balls and n bins.

• **Theorem:** when the threshold is near integer multiples of $\frac{m}{n}$, there is a high disparity in voting power (w.h.p.)

• **Theorem:** when the threshold is well-away from integer multiples of $\frac{m}{n}$, all agents have nearly identical voting power (w.h.p.)
Balls and Bins: Exponential

• There are m voters. A voter votes for player i w.p. \( p^i + 1 \)
• The probability of high-index players getting votes is extremely low. Most votes go to a few candidates.
• **Theorem:** if weights are drawn from an exponential balls-and–bins distribution, then with high probability, the resulting weights are **super-increasing**
• A vector of weights \((w_1, \ldots, w_n)\) is called **super-increasing** if

\[
\forall i \in N: \ w_i \geq \sum_{j<i} w_j
\]
Balls and Bins: Exponential

• In order to study the Shapley value in the Balls and Bins exponential case, it suffices to understand super-increasing sequences of weights.
• Suppose that weights are 1, 2, 4, 8, ... , \(2^{n-1}\) \((w_i = 2^{i-1})\)
• Let us observe the (beautiful) graph that results.
Super-Increasing Weights

• $\beta(S) = \sum_{i \in S} 2^{i-1}$: the binary representation of $S$
• $A(q)$: the minimal set $S \subseteq N$ such that $w(S) \geq q$
• **Claim**: if the weights are super-increasing, then
  $$\varphi^w_i(q) = \varphi^\beta_i(\beta(A(q)))$$
• the Shapley value when the threshold is $q$ equals the Shapley value when the weights are powers of 2, and the threshold is $\beta(A(q))$
• Computing the Shapley value for super-increasing weights boils down to computing it for powers of 2!
• Using this claim, we obtain a **closed-form formula** of the SV when the weights are super-increasing.
Conclusion

• Computation: generally, computing the Shapley value (and the Banzhaf value) is \#P complete (counting complexity)

• It is easy when we know that the weights are not too large (pseudopolynomial time)

• It is easy to approximate them through random sampling in the case of simple games.
Further Reading

• Chalkiadakis et al. “Computational Aspects of Cooperative Game Theory”
• Zuckerman et al. “Manipulating the Quota in Weighted Voting Games” (JAIR’12)
• Zick et al. “The Shapley Value as a Function of the Quota in Weighted Voting Games” (IJCAI’11)
• Zick “On Random Quotas and Proportional Representation in Weighted Voting Games” (IJCAI’13)
• Oren et al. “On the Effects of Priors in Weighted Voting Games” (COMSOC’14)