CMU 15-896
Social choice 1: The basics

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Social choice theory

- A mathematical theory that deals with aggregation of individual preferences
- Origins in ancient Greece
- Formal foundations: 18th Century (Condorcet and Borda)
- 19th Century: Charles Dodgson
- 20th Century: Nobel prizes to Arrow and Sen
The voting model

• Set of voters $N = \{1, \ldots, n\}$
• Set of alternatives $A, |A| = m$
• Each voter has a ranking over the alternatives
• Preference profile = collection of all voters’ rankings

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<thead>
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<th>1</th>
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Voting rules

• Voting rule = function from preference profiles to alternatives that specifies the winner of the election

• Plurality
  o Each voter awards one point to top alternative
  o Alternative with most points wins
  o Used in almost all political elections
More Voting Rules

- Borda count
  - Each voter awards $m - k$ points to alternative ranked $k$’th
  - Alternative with most points wins
  - Proposed in the 18th Century by the chevalier de Borda
  - Used for elections to the national assembly of Slovenia
  - Similar to rule used in the Eurovision song contest

Lordi, Eurovision 2006 winners
More voting rules

• Positional scoring rules
  o Defined by vector \((s_1, \ldots, s_m)\)
  o Plurality = \((1,0,\ldots,0)\), Borda = \((m-1, m-2, \ldots, 0)\)

• \(x\) beats \(y\) in a pairwise election if the majority of voters prefer \(x\) to \(y\)

• Plurality with runoff
  o First round: two alternatives with highest plurality scores survive
  o Second round: pairwise election between these two alternatives
More voting rules

• Single Transferable vote (STV)
  o \( m - 1 \) rounds
  o In each round, alternative with least plurality votes is eliminated
  o Alternative left standing is the winner
  o Used in Ireland, Malta, Australia, and New Zealand (and Cambridge, MA)
### STV: Example

<table>
<thead>
<tr>
<th>2 voters</th>
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<th>1 voter</th>
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Social choice axioms

• How do we choose among the different voting rules? Via desirable properties!
• Majority consistency = if a majority of voters rank alternative $x$ first, then $x$ should be the winner

Which of the rules we talked about is not majority consistent?
Marquis de Condorcet

- 18th Century French Mathematician, philosopher, political scientist
- One of the leaders of the French revolution
- After the revolution became a fugitive
- His cover was blown and he died mysteriously in prison
Condorcet winner

• Recall: $x$ beats $y$ in a pairwise election if a majority of voters rank $x$ above $y$

• Condorcet winner beats every other alternative in pairwise election

• Condorcet paradox = cycle in majority preferences
Condorcet consistency

• Condorcet consistency = select a Condorcet winner if one exists

Which of the rules we talked about is Condorcet consistent?
Condorcet consistency

Poll: What is the relation between majority consistency and Condorcet consistency?

3. Equivalent
4. Incomparable
More voting rules

• Copeland
  o Alternative’s score is \#alternatives it beats in pairwise elections
  o Why does Copeland satisfy the Condorcet criterion?

• Maximin
  o Score of \( x \) is \( \min_y |\{i \in N: x >_i y\}| \)
  o Why does Maximin satisfy the Condorcet criterion?
Application: Web Search

• Generalized Condorcet: if there is a partition $X, Y$ of $A$ such that a majority prefers every $x \in X$ to every $y \in Y$, then $X$ is ranked above $Y$

• Assumption: spam website identified by a majority of search engines

• When aggregating results from different search engines, spam websites will be ranked last [Dwork et al., WWW 2001]
APPLICATION: Web Search

x
a
b
z
y

a
y
b
z
x

b
z
a
x
y

a
b
z
x
y

overall
Dodgson’s Rule

• Distance function between profiles: #swaps between adjacent candidates

• Dodgson score of $x =$ the min distance from a profile where $x$ is a Condorcet winner

• Dodgson’s rule: select candidate that minimizes Dodgson score

• The problem of computing the Dodgson score is NP-complete!
Dodgson Unleashed

Voter 1
- a
- b
- c
- d
- e

Voter 2
- b
- a
- c
- d
- e

Voter 3
- e
- b
- c
- d
- a

Voter 4
- e
- c
- d
- b
- a

Voter 5
- b
- e
- d
- c
- a
Awesome example

• Plurality: \( a \)
• Borda: \( b \)
• Condorcet winner: \( c \)
• STV: \( d \)
• Plurality with runoff: \( e \)

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<th>33 voters</th>
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<th>3 voters</th>
<th>8 voters</th>
<th>18 voters</th>
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Is social choice practical?

• UK referendum: Choose between plurality and STV as a method for electing MPs
• Academics agreed STV is better...
• ... but STV seen as beneficial to the hated Nick Clegg
• Hard to change political elections!
Computational social choice

• However:
  o in human computation systems...
  o in multiagent systems...
    the designer is free to employ any voting rule!

• Computational social choice focuses on positive results through computational thinking
Example: Robobees

- Robobees need to decide on a joint plan (alternative)
- Many possible plans
- Each robobee (agent) has a numerical evaluation (utility) for each alternative
- Want to maximize sum of utilities = social welfare
- Communication is restricted
Example: Robobees

• Approach 1: communicate utilities
  o May be infeasible
• Approach 2: each agent votes for favorite alternative (plurality)
  o $\log m$ bits per agent
  o May select a bad alternative
Example: Robobees

- Approach 3: each agent votes for an alternative with probability proportional to its utility
- Theorem [Caragiannis & P 2011]: if $n = \omega(m\log m)$ then this approach gives almost optimal social welfare in expectation
Example: Pnyx

A powerful & user-friendly preference aggregation tool

<table>
<thead>
<tr>
<th></th>
<th>Most preferred alternative</th>
<th>Approved alternatives</th>
<th>Linear rankins</th>
<th>Rankings with ties</th>
<th>Pairwise comparisons</th>
</tr>
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<tr>
<td>Unique winner</td>
<td>Plurality rule</td>
<td>Approval voting</td>
<td>Borda's rule</td>
<td>Bucket Borda's rule</td>
<td>Young's generalization of Borda's rule</td>
</tr>
<tr>
<td>Lottery</td>
<td>Random dictatorship</td>
<td>Nash's rule</td>
<td>Maximal lotteries</td>
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<td>Ranking without ties</td>
<td>Plurality scores</td>
<td>Approval voting scores</td>
<td>Kemeny's rule</td>
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