15896 Spring 2016 Homework #3: Noncooperative Game Theory

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Due: 4/7/2016 11:59pm

Rules:

- I encourage you to work with a friend on the problems. If you do work with a friend, please submit a single solution.
- It's fine to look up a complicated sum or inequality or whatever, but don't look up an entire solution. In particular, the solutions to many of the problems that I give can be found in papers, but, needless to say, you should avoid reading the proof if you come across the relevant paper. If for some reason you did see the solution before working it out yourself, please say so in your solution.
- Please prepare a pdf with your solution and send it to me (Ariel) by email.

Problems:

1. **[15 points]** In the Cycle Cover problem, we are given a directed graph and two integers k, t; we are asked whether it is possible to cover at least t vertices with disjoint cycles of length at most k. We stated in class (lecture 14, slide 10) that the Cycle Cover problem is NP-hard when there is a given upper bound k on the length of cycles. Show that if there is no such upper bound then the problem can be solved in polynomial time.

   Hint: You may rely on the fact that a maximum weight perfect matching in a bipartite graph can be computed in polynomial time.

2. Consider a 2-player normal-form game with n pure strategies per player, where all utilities are in $[0, 1]$. We say that a (possibly mixed) strategy profile (x_1, x_2) is an ϵ-Nash equilibrium if each player cannot gain more than ϵ by deviating. That is, for each $i \in \{1, 2\}$ and $x'_i \in \Delta(S)$, $u_i(x'_i, x_{-i}) \leq u_i(x_i, x_{-i}) + \epsilon$.

   (a) **[30 points]** Prove that there exists an ϵ-Nash equilibrium in which each player has positive probability on at most $O(\frac{1}{\epsilon^2} \log n)$ strategies.

   Hint: You may rely on the existence of a Nash equilibrium. You will also want somewhere to use Hoeffding's inequality, which says that if $X = \frac{1}{m} \sum_{k=1}^{m} X_k$, where the X_k are i.i.d. $[0, 1]$-valued random variables, then

   $$\Pr[|X - \mathbb{E}[X]| > \epsilon] \leq 2e^{-2m\epsilon^2}.$$

(b) **[10 points]** Design an algorithm for computing an ϵ-Nash equilibrium that runs in time $n^{O(\frac{1}{\epsilon^2}\log n)}$.

3. Consider the following scheduling game. The players $N = \{1, \ldots, n\}$ are associated with tasks, each with weight w_i. There is also a set M of m machines. Each player chooses a machine to place his task on, that is, the strategy space of each agent is M. A strategy profile induces an assignment $A : N \to M$ of agents (or tasks) to machines; the *cost* of player i is the total load on the machine to which i is assigned — $\ell_{A(i)} = \sum_{j \in N:\ A(j)=A(i)} w_j$. Our objective function is the *makespan*, which is the maximum load on any machine: $\text{cost}(A) = \max_{\mu \in M} \ell_\mu$. It is known that scheduling games always have pure Nash equilibria.

   (a) **[30 points]** Let G be a scheduling game with n tasks of weight $w_1, \ldots, w_n$, and m machines. Let $A : N \to M$ be a Nash equilibrium assignment. Then

   $$\text{cost}(A) \leq \left(2 - \frac{2}{m+1}\right) \cdot \text{opt}(G).$$

   That is, the (pure) price of anarchy is at most $2 - 2/(m+1)$.

   (b) **[15 points]** Prove that the upper bound of part (a) is tight, by constructing an appropriate family of scheduling games for each $m \in \mathbb{N}$.