1. Consider the cake cutting problem with \( n \) players and valuation functions \( V_1, \ldots, V_n \) satisfying additivity, normalization, and divisibility (lecture 6, slide 3). Denote the social welfare of an allocation \( A \) by \( \text{sw}(A) = \sum_{i=1}^{n} V_i(A_i) \).

(a) [25 points] Show that for all valuation functions \( V_1, \ldots, V_n \),

\[
\frac{\max\{\text{sw}(A) : A \text{ is an allocation of the cake}\}}{\max\{\text{sw}(A) : A \text{ is a proportional allocation of the cake}\}} = O(\sqrt{n}).
\]

(b) [10 points] Give a family of examples of \( V_1, \ldots, V_n \) (one example for each value of \( n \)) such that

\[
\frac{\max\{\text{sw}(A) : A \text{ is an allocation of the cake}\}}{\max\{\text{sw}(A) : A \text{ is a proportional allocation of the cake}\}} = \Omega(\sqrt{n}).
\]

2. Consider a set of players with Leontief preferences. In class we discussed the dominant resource fairness (DRF) mechanism, which equalizes the dominant shares. Consider instead the asset fairness mechanism, which equalizes the sum of shares subject to allocating proportionally to the reported demands. In other words, it outputs an allocation \( A \) that is proportional to the demands and satisfies \( \sum_r A_{ir} = \sum_r A_{jr} \) for every two players \( i, j \in N \).

(a) [5 points] Prove/disprove: Asset fairness is envy free.
(b) [5 points] Prove/disprove: Asset fairness is proportional, that is, for all \( i \in N, u_i(A_i) \geq u_i((1/n, 1/n, \ldots, 1/n)) \).

**Note**: Don’t confuse the proportionality property, which is a fairness property (and the focus of part (b)), with being proportional to the demands, which is also known as non-wastefulness.

(c) [15 points] Prove/disprove: Asset fairness is strategyproof.

3. We say that a cooperative game \( G = (N, v) \) is convex if for any \( T \subseteq N, S \subseteq T, \) and any \( i \in N \setminus T \) we have \( v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T) \).

Given a permutation \( \sigma \in \Pi(N) \), we let \( \vec{x}(\sigma) \) be the imputation defined by

\[
x_i(\sigma) = v(P_i(\sigma) \cup \{i\}) - v(P_i(\sigma)),
\]

where \( P_i(\sigma) \) is the set of predecessors of \( i \) in \( \sigma \). In words, the payoff to \( i \) is the marginal contribution of \( i \) to his predecessors. Let us define \( \text{Conv}(\vec{x}(\sigma))_{\sigma \in \Pi(N)} \) to be the convex hull of the vectors \( \vec{x}(\sigma) \).

(a) [10 points] Show that if \( G = (N, v) \) is a convex game, then

\[
\text{Conv}(\vec{x}(\sigma))_{\sigma \in \Pi(N)} \subseteq \text{Core}(G).
\]

(b) [5 points] Conclude that the Shapley value of a convex game is in the core.

4. [25 points] We proved in class (lecture 13, slides 5–7) that when there are at least two players, no deterministic strategyproof kidney exchange mechanism can provide an \( \alpha \)-approximation for \( \alpha < 2 \). Show that no randomized strategyproof kidney exchange mechanism can provide an \( \alpha \)-approximation for \( \alpha < 6/5 \).