15896 Spring 2016 Homework #2:
Fair Division, Cooperative Games, and Kidney Exchange

Ariel Procaccia (arielpro@cs.cmu.edu)

Due: 3/14/2016 11:59pm

Rules:

- I encourage you to work with a friend on the problems. If you do work with a friend, please submit a single solution.
- It's fine to look up a complicated sum or inequality or whatever, but don't look up an entire solution. In particular, the solutions to many of the problems that I give can be found in papers, but, needless to say, you should avoid reading the proof if you come across the relevant paper. If for some reason you did see the solution before working it out yourself, please say so in your solution.
- Please prepare a pdf with your solution and send it to me (Ariel) by email.

Problems:

1. Consider the cake cutting problem with n players and valuation functions $V_1, \ldots, V_n$ satisfying additivity, normalization, and divisibility (lecture 6, slide 3). Denote the *social welfare* of an allocation $\mathbf{A}$ by $\text{sw}(\mathbf{A}) = \sum_{i=1}^{n} V_i(A_i)$.

   (a) **[25 points]** Show that for all valuation functions $V_1, \ldots, V_n$,

   $$\frac{\max\{\text{sw}(\mathbf{A}) : \ \mathbf{A} \text{ is an allocation of the cake}\}}{\max\{\text{sw}(\mathbf{A}) : \ \mathbf{A} \text{ is a } \textit{proportional} \text{ allocation of the cake}\}} = O(\sqrt{n}).$$

   (b) **[10 points]** Give a family of examples of $V_1, \ldots, V_n$ (one example for each value of n) such that

   $$\frac{\max\{\text{sw}(\mathbf{A}) : \ \mathbf{A} \text{ is an allocation of the cake}\}}{\max\{\text{sw}(\mathbf{A}) : \ \mathbf{A} \text{ is a } \textit{proportional} \text{ allocation of the cake}\}} = \Omega(\sqrt{n}).$$

2. Consider a set of players with Leontief preferences. In class we discussed the *dominant resource fairness (DRF)* mechanism, which equalizes the dominant shares. Consider instead the *asset fairness* mechanism, which equalizes the sum of shares subject to allocating proportionally to the reported demands. In other words, it outputs an allocation $\mathbf{A}$ that is proportional to the demands and satisfies $\sum_r A_{ir} = \sum_r A_{jr}$ for every two players $i, j \in N$.

   (a) **[5 points]** Prove/disprove: Asset fairness is envy free.

(b) **[5 points]** Prove/disprove: Asset fairness is proportional, that is, for all $i \in N$, $u_i(\mathbf{A}_i) \geq u_i((1/n, 1/n, \ldots, 1/n))$.

Note: Don't confuse the proportionality property, which is a fairness property (and the focus of part (b)), with being proportional to the demands, which is also known as *non-wastefulness*.

(c) **[15 points]** Prove/disprove: Asset fairness is strategyproof.

3. We say that a cooperative game $\mathcal{G} = (N, v)$ is *convex* if for any $T \subsetneq N$, $S \subseteq T$, and any $i \in N \setminus T$ we have $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$.

   Given a permutation $\sigma \in \Pi(N)$, we let $\vec{x}(\sigma)$ be the imputation defined by

$$x_i(\sigma) = v(P_i(\sigma) \cup \{i\}) - v(P_i(\sigma)),$$

   where $P_i(\sigma)$ is the set of predecessors of i in σ. In words, the payoff to i is the marginal contribution of i to his predecessors. Let us define $Conv\,(\vec{x}(\sigma))_{\sigma \in \Pi(N)}$ to be the convex hull of the vectors $\vec{x}(\sigma)$.

   (a) **[10 points]** Show that if $\mathcal{G} = (N, v)$ is a convex game, then

$$Conv\,(\vec{x}(\sigma))_{\sigma \in \Pi(N)} \subseteq Core(\mathcal{G}).$$

   (b) **[5 points]** Conclude that the Shapley value of a convex game is in the core.

4. **[25 points]** We proved in class (lecture 13, slides 5–7) that when there are at least two players, no deterministic strategyproof kidney exchange mechanism can provide an α-approximation for $\alpha < 2$. Show that no *randomized* strategyproof kidney exchange mechanism can provide an α-approximation for $\alpha < 6/5$.