15896 Spring 2016 Homework #1: Social Choice

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Due: 2/4/2015 11:59pm

Rules:

- I encourage you to work with a friend on the problems. If you do work with a friend, please submit a single solution.
- It's fine to look up a complicated sum or inequality or whatever, but don't look up an entire solution. In particular, the solutions to many of the problems that I give can be found in papers, but, needless to say, you should avoid reading the proof if you come across the relevant paper. If for some reason you did see the solution before working it out yourself, please say so in your solution.
- Please prepare a pdf with your solution and send it to me (Ariel) by email.

Problems:

1. We discussed several notions of monotonicity. Here is the most common one: if $f(\vec{\succ}) = a$ and $\vec{\succ}'$ is a profile such that (i) $[a \succ_i x \;\Rightarrow\; a \succ_i' x]$ for all $x \in A$ and $i \in N$, and (ii) $[x \succ_i y \;\Leftrightarrow\; x \succ_i' y]$ for all $x, y \in A \setminus \{a\}$ and $i \in N$, then $f(\vec{\succ}') = a$. Informally, if you push a upwards and everything else remains the same, a stays the winner. Monotonicity is considered a desirable property (much like other axioms that we discussed); it is easy to see that it is satisfied by the rules that we discussed in class, except:

   (a) [**5 points**] Show that STV is nonmonotonic by giving a counterexample.

   (b) [**20 points**] Show that Dodgson's rule is nonmonotonic by giving a counterexample.

2. Prove the following statements:

   (a) [**10 points**] Let f be a strategyproof voting rule, $\vec{\succ}$ be a preference profile, and $f(\vec{\succ}) = a$. If $\vec{\succ}'$ is a profile such that $[a \succ_i x \;\Rightarrow\; a \succ_i' x]$ for all $x \in A$ and $i \in N$, then $f(\vec{\succ}') = a$. (Note that this is a much stronger property than the monotonicity property in Problem 1: if a is pushed upwards then it stays the winner even when other alternatives move as well.)

   (b) [**10 points**] Let f be a strategyproof and onto voting rule. Furthermore, let $\vec{\succ}$ be a preference profile and $a, b \in A$ such that $a \succ_i b$ for all $i \in N$. Then $f(\vec{\succ}) \neq b$. **Hint**: use part (a).

(c) **[10 points]** Let m be the number of alternatives and n be the number of voters, and assume that $m \geq 3$ and $m \geq n$. Furthrmore, let f be a strategyproof and neutral voting rule. Then f is dictatorial. **Important note:** This is the Gibbard-Satterthwaite Theorem for the special case of $m \geq n$ and a neutral voting rule. There are many proofs of the full version of the Gibbard-Satterthwaite Theorem; *here the task is specifically to formalize the proof sketch we did in Lecture 2.*

3. **[20 points]** Given a positional scoring rule parametrized by $s_1, \ldots, s_m$, where m is the number of alternatives, and a preference profile $\vec{\succ}$, let $\mathrm{sc}(\vec{\succ}, x)$ be the score of x when the preferences are $\vec{\succ}$. Design a strategyproof randomized voting rule that on every profile $\vec{\succ}$ outputs a distribution over alternatives such that the expected score of the winner is at least $\frac{\max_{x \in A} \mathrm{sc}(\vec{\succ}, x)}{O(\sqrt{m})}$, where m is the number of alternatives.

4. Let $N = \{1, \ldots, n\}$ be a set of *agents* (vertices of a graph), $n \geq 2$, and let $k \in \{1, \ldots, n-1\}$. A k-selection mechanism is called *group impartial (GI)* if there is no coalition of agents that can all gain from jointly misreporting their outgoing edges. Formally, f is group impartial if and only if for every $S \subseteq N$ and every pair of graphs G, G' that differ only in the outgoing edges of the agents in S, there exists $i \in S$ such that $\Pr[i \in f(G')] \leq \Pr[i \in f(G)]$, that is, the probability that i is selected by f given G' is at most the probability that i is selected by f given G.

   (a) **[5 points]** Give a randomized GI k-selection mechanism with an approximation ratio of n/k.

   (b) **[20 points]** Show that no randomized GI k-selection mechanism can provide an approximation ratio smaller than $(n-1)/k$.

   (c) **[BONUS 20 points]** Close the gap between the upper bound and lower bound (this is a minor open problem).