15896 Spring 2015 Homework #2: Cooperative Games and Fair Division

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Due: 3/5/2015 1:30pm

Rules:

- You may work with a friend on the problems if you want. If you do work with a friend, please submit a single solution.
- If you've seen a problem before (some of them are "famous"), then say that in your solution. Also, if you use any sources other than the AGT book, write that down too. **Please don't deliberately search for solutions online!** It's fine to look up a complicated sum or inequality or whatever, but don't look up an entire solution.
- Please prepare a pdf with your solution and send it to Ariel by email.

Problems:

1. We say that a cooperative game $\mathcal{G} = (N, v)$ is *convex* if for any $T \subsetneq N$, $S \subseteq T$, and any $i \in N \setminus T$ we have $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$.

   Given a permutation $\sigma \in \Pi(N)$, we let $\vec{x}(\sigma)$ be the imputation defined by $x_i(\sigma) = v(P_i(\sigma) \cup \{i\}) - v(P_i(\sigma))$, where $P_i(\sigma)$ is the set of predecessors of i in σ. In words, the payoff to i is the marginal contribution of i to his predecessors. Let us define $Conv\left(\vec{x}(\sigma)\right)_{\sigma \in \Pi(N)}$ to be the convex hull of the vectors $\vec{x}(\sigma)$.

   (a) **[15 points]** Show that if $\mathcal{G} = (N, v)$ is a convex game, then

   $$Conv\left(\vec{x}(\sigma)\right)_{\sigma \in \Pi(N)} \subseteq Core(\mathcal{G}).$$

   (b) **[5 points]** Conclude that the Shapley value of a convex game is in the core.

2. **[25 points]** A solution to the bankruptcy problem is a function f whose input is a bankruptcy problem (a vector $\vec{c} = (c_1, \ldots, c_n) \in \mathbb{R}^n_+$ and $E > 0$ such that $\sum_{i=1}^n c_i > E$), and whose output is a vector $\vec{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n_+$ such that $\sum_{i \in N} x_i = E$. A solution f is consistent with the contested garment solution if for every two claimants $i, j \in N$, $f_i(c_1, \ldots, c_n; E) = CG_i(c_i, c_j; x_i + x_j)$. Here, $CG_i(c_1, c_2; E)$ is the solution to the contested garment problem (two claimant variant of the bankruptcy problem) discussed in class. We have shown that there is a unique solution ϕ that is consistent with the contested garment problem; it is the

one described by the hydraulic machinery presented in class, which is based on the law of communicating vessels.[1]

Give a closed-form formula for ϕ, and explain how you derived it.

Guidance: The formula can have different cases, depending on the value of E. In addition, the explanation can be intuitive — a formal proof is not required.

3. Consider the following class of valuation functions over a cake. For every player i, the function V_i is represented by points $0 = t_1 < t_2 < \cdots < t_{K-1} < t_K = 1$, and values $v_1, \ldots, v_K$ (where for simplicity we use the same K for all players) such that $V_i(t_{s-1}, t_s) = v_s$ for all $s = 2, \ldots, K$. Moreover, it is assumed that the value V_i is uniformly distributed over each interval $[t_{s-1}, t_s]$. For example, if $I \subset [t_{s-1}, t_s]$ and $|I| = \lambda(t_s - t_{s-1})$ then $V_i(I) = \lambda v_s$.

   (a) **[10 points]** Give an $O(n^3K^3)$ algorithm that computes an envy-free allocation given valuation functions that are represented as above.

   Note: Your running time is actually likely to be lower than the stated upper bound; I just don't want the running time to be an issue (because the algorithm I have in mind is very simple), but at the same time I'm trying to prevent you from using linear programming for part (a).

   (b) **[25 points]** Given valuations functions that are represented as above, formulate a linear program (LP) that computes an allocation that maximizes social welfare (i.e., $\sum_{i \in N} V_i(A_i)$) among all envy-free allocations. The number of variables and constraints in your LP should be polynomial in the representation of the input. (This shows that in this setting a welfare-maximizing envy-free allocation can be computed in polynomial time.)

   Note: I imagine that almost all of you know what a linear program is. If you've never seen one, please consult Wikipedia. If the Wikipedia page is unclear, please ask me for an explanation by email or in person.

4. Consider a set of players with Leontief preferences. In class we discussed the *dominant resource fairness (DRF)* mechanism, which equalizes the dominant shares. Consider instead the *asset fairness* mechanism, which equalizes the sum of shares subject to allocating proportionally to the reported demands. In other words, it outputs an allocation $\mathbf{A}$ that is proportional to the demands and satisfies $\sum_r A_{ir} = \sum_r A_{jr}$ for every two players $i, j \in N$.

   (a) **[10 points]** Prove/disprove: Asset fairness is envy free.

   (b) **[10 points]** Prove/disprove: Asset fairness is proportional, that is, for all $i \in N$, $u_i(\mathbf{A}_i) \geq u_i((1/n, 1/n, \ldots, 1/n))$.

   Note: Don't confuse the proportionality property, which is a fairness property (and the focus of part (b)), with being proportional to the demands, which is also known as *non-wastefulness.*

[1] We thank Itamar Procaccia (Ariel's dad) for providing us with the proper English term for this physical law!