15896 Spring 2015 Homework #1: Social Choice

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Due: 2/5/2015 1:30pm

Rules:

- You may work with a friend on the problems if you want. If you do work with a friend, please submit a single solution.
- If you've seen a problem before (sometimes I'll give problems that are "famous"), then say that in your solution. It won't affect your score, I just want to know. Also, if you use any sources other than the AGT book, write that down too. It's fine to look up a complicated sum or inequality or whatever, but don't look up an entire solution.
- Please prepare a pdf with your solution and send it to me (Ariel) by email.

Problems:

1. We discussed several notions of monotonicity. Here is the most common one: if $f(\vec{\succ}) = a$ and $\vec{\succ}'$ is a profile such that (i) $[a \succ_i x \;\Rightarrow\; a \succ'_i x]$ for all $x \in A$ and $i \in N$, and (ii) $[x \succ_i y \;\Leftrightarrow\; x \succ'_i y]$ for all $x, y \in A \setminus \{a\}$ and $i \in N$, then $f(\vec{\succ}') = a$. Informally, if you push a upwards and everything else remains the same, a stays the winner. Monotonicity is considered a desirable property (much like other axioms that we discussed); it is easy to see that it is satisfied by the rules that we discussed in class, except:

   (a) [**15 points**] Show that STV is nonmonotonic by giving a counterexample.

   (b) [**25 points**] Show that Dodgson's rule is nonmonotonic by giving a counterexample.

2. Prove the following statements:

   (a) [**10 points**] Let f be a strategyproof voting rule, $\vec{\succ}$ be a preference profile, and $f(\vec{\succ}) = a$. If $\vec{\succ}'$ is a profile such that $[a \succ_i x \;\Rightarrow\; a \succ'_i x]$ for all $x \in A$ and $i \in N$, then $f(\vec{\succ}') = a$. (Note that this is a much stronger property than the monotonicity property in Problem 1: if a is pushed upwards then it stays the winner even when other alternatives move as well.)

   (b) [**10 points**] Let f be a strategyproof and onto voting rule. Furthermore, let $\vec{\succ}$ be a preference profile and $a, b \in A$ such that $a \succ_i b$ for all $i \in N$. Then $f(\vec{\succ}) \neq b$. **Hint**: use part (a).

(c) **[15 points]** Let m be the number of alternatives and n be the number of voters, and assume that $m \geq 3$ and $m \geq n$. Furthrmore, let f be a strategyproof and neutral voting rule. Then f is dictatorial. **Important note:** This is the Gibbard-Satterthwaite Theorem for the special case of $m \geq n$ and a neutral voting rule. There are many proofs of the full version of the Gibbard-Satterthwaite Theorem; *here the task is specifically to formalize the proof sketch we did in Lecture 2.*

3. **[25 points]** Given a preference profile $\vec{\succ}$, let $\text{sc}(\vec{\succ}, x)$ be the plurality score of x when the preferences are $\vec{\succ}$ (i.e., the number of voters who rank x first). Design a strategyproof randomized voting rule that on every profile $\vec{\succ}$ outputs a distribution over alternatives such that the expected score of the winner is at least $\frac{\max_{x \in A} \text{sc}(\vec{\succ}, x)}{O(\sqrt{m})}$, where m is the number of alternatives (of course you need to formally establish strategyproofness and this approximation guarantee).