15896 Spring 2015 Homework #1: Social Choice

Ariel Procaccia (arielpro@cs.cmu.edu)

Due: 2/5/2015 1:30pm

Rules:

• You may work with a friend on the problems if you want. If you do work with a friend, please submit a single solution.

• If you’ve seen a problem before (sometimes I’ll give problems that are “famous”), then say that in your solution. It won’t affect your score, I just want to know. Also, if you use any sources other than the AGT book, write that down too. It’s fine to look up a complicated sum or inequality or whatever, but don’t look up an entire solution.

• Please prepare a pdf with your solution and send it to me (Ariel) by email.

Problems:

1. We discussed several notions of monotonicity. Here is the most common one: if \( f(\succ) = a \) and \( \succ' \) is a profile such that (i) \( a \succ_i x \Rightarrow a \succ'_i x \) for all \( x \in A \) and \( i \in N \), and (ii) \( x \succ_i y \Leftrightarrow x \succ'_i y \) for all \( x, y \in A \setminus \{a\} \) and \( i \in N \), then \( f(\succ') = a \). Informally, if you push \( a \) upwards and everything else remains the same, \( a \) stays the winner. Monotonicity is considered a desirable property (much like other axioms that we discussed); it is easy to see that it is satisfied by the rules that we discussed in class, except:

   (a) [15 points] Show that STV is nonmonotonic by giving a counterexample.

   (b) [25 points] Show that Dodgson’s rule is nonmonotonic by giving a counterexample.

2. Prove the following statements:

   (a) [10 points] Let \( f \) be a strategyproof voting rule, \( \succ \) be a preference profile, and \( f(\succ) = a \). If \( \succ' \) is a profile such that \( [a \succ_i x \Rightarrow a \succ'_i x] \) for all \( x \in A \) and \( i \in N \), then \( f(\succ') = a \). (Note that this is a much stronger property than the monotonicity property in Problem 1: if \( a \) is pushed upwards then it stays the winner even when other alternatives move as well.)

   (b) [10 points] Let \( f \) be a strategyproof and onto voting rule. Furthermore, let \( \succ \) be a preference profile and \( a, b \in A \) such that \( a \succ_i b \) for all \( i \in N \). Then \( f(\succ) \neq b \). **Hint:** use part (a).
(c) [15 points] Let \( m \) be the number of alternatives and \( n \) be the number of voters, and assume that \( m \geq 3 \) and \( m \geq n \). Furthermore, let \( f \) be a strategyproof and neutral voting rule. Then \( f \) is dictatorial. **Important note:** This is the Gibbard-Satterthwaite Theorem for the special case of \( m \geq n \) and a neutral voting rule. There are many proofs of the full version of the Gibbard-Satterthwaite Theorem; here the task is specifically to formalize the proof sketch we did in Lecture 2.

3. [25 points] Given a preference profile \( \succ \), let \( \text{sc}(\succ, x) \) be the plurality score of \( x \) when the preferences are \( \succ \) (i.e., the number of voters who rank \( x \) first). Design a strategyproof randomized voting rule that on every profile \( \succ \) outputs a distribution over alternatives such that the expected score of the winner is at least \( \frac{\max_{x \in A} \text{sc}(\succ, x)}{O(\sqrt{m})} \), where \( m \) is the number of alternatives (of course you need to formally establish strategyproofness and this approximation guarantee).