Games are an abstraction
Aim to identify most important decisions of participants and main factors in payoffs.
- But there are always low-order unmodeled factors or variables.
- And some players whose incentives not modeled well at all.

High-level question: How much impact can these issues have on dynamics in the system?

Games are an abstraction
We’d like to think that if we get people into a good equilibrium, and players are selfish, reasonably myopic, etc, then behavior will stay there.

But what if there are small fluctuations in underlying cost/payoff structure? Could they cause natural dynamics to spin out of control?

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Can small fluctuations or a few unpredictable players act unpredictably?

Or what about a few players acting unpredictably?
**High-level question**

A few ways this could happen:
- Small changes cause good equilibria to disappear, only bad ones left. (economy?)
- Bad behavior by a few players causes pain for all (nukes)
- Neither of above, but instead through more subtle interaction with dynamics...

Can small fluctuations or a few unpredictable players cause natural dynamics to get system into a high-cost state? Focus here on this last issue.

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**Focus of this work**

Games with the following properties:
- Potential games, best/better-response dynamics.
- Small gap between potential and social cost.
  - Without fluctuations, can walk away from good state even if not an equilibrium: gap bounds possible increase.
  - Single perturbation can't make dynamics do bad things.

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**Focus of this work**

Games with the following properties:
- Potential games, best/better-response dynamics.
- Small gap between potential and social cost.
- No individual player can influence total cost of others by too much. (With Byzantine players, will define social cost as sum of costs to others)

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**Focus of this work**

Games with the following properties:
- Potential games, best/better-response dynamics.
- Example: set-cover games.
  - n players, m resources, with costs $c_1, ..., c_m$.
  - Each player allowed to use some resources, not others.
  - Each player chooses some allowable resource.
  - Players split cost with all others choosing same one.

$$\text{cost}(S) = \sum_{i=1}^{m} c_i$$

$$\phi(S) = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}$$

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**Model**

- Players follow best (or better) response dynamics.
- Costs of resources can fluctuate between moves: $c_i^{t} \in [c_i, c_i(1+\varepsilon)]$
- Alternatively, one/few Byzantine players who move between time steps
- Play begins in a low-cost state.
- How bad can things get?

**Price-of-Uncertainty($\varepsilon$)** of game = maximum ratio of eventual social cost to initial cost.
Model

- Players follow best (or better) response dynamics.
- Costs of resources can fluctuate between moves: \( c^i_t \in [c_i, c_i(1+\epsilon)] \)

Price-of-Uncertainty(\(\epsilon\)) of game = maximum ratio of eventual social cost to initial cost.

Set-cover games

- \( n \) players, \( m \) resources. Each player chooses one of allowable resources. Players split cost with all others choosing same one.
- Price of anarchy = \( n \)
- Potential \( \Phi \in [\text{cost, cost-log(n)}] \)

Main results

General fair-cost-sharing games:
- If many players for each \((s_i,t_i)\) pair \((n_i = \Omega(m))\), then \( \text{PoU} = O(1) \) even for constant \( \epsilon > 0 \).
- Open for general number of players.

Matroid congestion games: (strategy sets are bases of matroid. E.g., set-cover where choose \( k \) resources)
- If \( \epsilon = O(1/nm) \) then \( \text{PoU} = O(\log n) \) for fair cost-sharing.
- In general, if \( \epsilon = O(1/nm) \) then \( \text{PoU} = O(GAP) \).

These require Best-Response. Better-resp not enough
Also results for \( \beta \)-nice games, job scheduling, …
Set-Cover games (lower bound)
A single Byzantine player can take state from a PNE of cost $O(OPT)$ to one of cost $\Omega(n \cdot OPT)$.

Set-Cover games (lower bound)
Two kinds of players:
- $n$ of Class I:
- $n-1$ of Class II:

Plus one Byzantine player...

- Implies bound for case that costs can fluctuate by factor of 2. More complicated ex for $\varepsilon \rightarrow 1/n^{1/2}$.

Set-Cover games (upper bound)
For upper bound, think of players in sets as a stack of chips.
- View $i$th position in stack $j$ as having cost $c_j/i$. Load chips with value equal to initial cost.
- When player moves from $j$ to $k$, move top chip. Cost of position goes up by at most $(1+\varepsilon)$.
- At most $mn$ different positions. So, following the path of any chip and removing loops, cost of final set is at most $(1+\varepsilon)^{nm}$ times its value.

So, if $\varepsilon = O(1/nm)$ then PoU = $O(\log n)$.

Matroid games
In matroid games, can think of each player as controlling a set of chips.
- Nice property of best response in matroids:
  - Can always order the move so that each individual chip is doing better-response.
  - Apply previous argument.
  - Fails for better-response.
    - Here, can get player to do kind of binary counting, bad even for exponentially-small $\varepsilon$.

Fair cost sharing in general graphs
If many players of each type, can also show best-response dynamics can't do badly.

Outline of argument:
- Hard to analyze cost of state directly, instead track upper bound $c^*(S_t) = \text{cost}(S_0 \cup ... \cup S_t)$.
  - $c^*$ changes at most $m$ times.
  - Many players of each type $\Rightarrow$ average cost of each is low compared to $c^* \Rightarrow$ each change to $c^*$ is small.

Because it can never be a BR move to switch to something of cost $\times (1+\varepsilon)$ times average for type.
PoU versus Price of Anarchy

Main focus: games with both good and bad equilibria.
- Can small fluctuations or single Byzantine player cause behavior to move from good to bad?
- Can also have cases where state can get worse even than worst equilibrium.
- Market-sharing: 2 vs log(n) even for best-response.

PoU versus Price of Anarchy

\(\beta\)-nice games [AAEMS-EC08]: incentives grow stronger as cost gets above \(\beta\) times optimal (typically \(\beta = \text{PoA}\)).
- \(2\Delta(S) > \text{cost}(S) - \beta \text{OPT}, \quad \Delta(S) = \sum_i \Delta_i(S)\).
- Here, at least can show state won't get above \(O(\beta)\) times optimal, even with substantial perturbation or many Byzantine players (random order).

Subsequent results

[Balcan, Constantin, Ehrlich]

- Set-cover games:
  - Upper bound with dependence only on \(m\), not \(n\).
  - Lower bound under random move-ordering.
- Consensus games:
  - Nearly-tight bounds on effect of \(\varepsilon\)-perturbations
  - Tight bounds on effect of \(B\) Byzantine players.

Summary and open problems

Looking at: when can small perturbations or a few bad players lead natural dynamics astray?
- When is it safe to turn your back?
- Upper/lower bounds for a number of classes of games.

Open problems:
- General case of fair cost-sharing games?
- Analyze time to failure for random fluctuations?
- Instance-based analysis?