CMU 15-896 Algorithms, Games, & Networks Spring 2013

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More on Nash equilibria: concepts, complexity, and algorithms

Your guide:

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[Readings: Ch. 2.1-2.4 of AGT book]

One more interesting game

"Ultimatum game":

- Two players "Splitter" and "Chooser"
- 3rd party puts $10 on table.
- Splitter gets to decide how to split between himself and Chooser.
- Chooser can accept or reject.
- If reject, money is burned.

One more interesting game

"Ultimatum game": E.g., with $4

Splitter: how much to offer chooser (columns); Chooser: how much to accept (rows)

	1	2	3
1	(1,3)	(2,2)	(3,1)
2	(0,0)	(2,2)	(3,1)
3	(0,0)	(0,0)	(3,1)

Stackelberg leader strategies

Strategy such that if you announce it and opponent best-responds to you, you are best off.

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Stackelberg leader strategies

Strategy such that if you announce it and opponent best-responds to you, you are best off.

Need not be a Nash equilibrium.

	Compete	Leave
Price high	(3,3)	(6,1)
Price low	(2,0)	(4,1)

Stackelberg leader strategies

Can solve efficiently. Say we're row player:

- For each column j, solve for p to maximize our expected gain s.t. j is best-response.
- Choose best.

	Compete	Leave
Price high	(3,3)	(6,1)
Price low	(2,0)	(4,1)

Hardness of computing Nash equilibria

Looking at 2-player n-action games.

2 types of results:

- NP-hardness for NE with special properties [Gilboa-Zemel] [Conitzer-Sandholm]
  - Is there one with payoff at least v for row?
  - Is there one using row #1?
  - Is there more than one?
  - …
- PPAD-hardness for finding any NE. [Chen-Deng][Daskalakis-Goldberg-Papadimitriou]

Hardness of computing Nash equilibria

NP-hardness for NE with special properties

Basic idea:

- Given 3-SAT formula F, create a game with one row for each literal, variable, & clause.
- Also a default attractor action f. $C = R^T$.
- Somehow set things up so that except for (f,f), all NE must correspond to satisfying assignments.

Hardness of computing Nash equilibria

NP-hardness for NE special properties

	L			V	C	f
L	$1+\frac{1}{n^2}$ / -1	-1 / $1+\frac{1}{n^2}$	$\ddots$	$-n^2$	$-n^2$	$-n^2$
V	x' / 2.0 / $\vdots$	x'	$\cdots$ / $\ddots$	$-n^2$	$-n^2$	$-n^2$
C	2.0 / 2.0	x' / 2.0	$\ddots$	$-n^2$	$-n^2$	$-n^2$
f		$1-\frac{1}{n^2}$		n^2	n^2	n^2

[$x' \approx -n$. These negative values for matches]

- (f,f) is default equilibrium.
- Unif over literals of satisfying assn are NE. Also mixture.

What about just finding some NE?

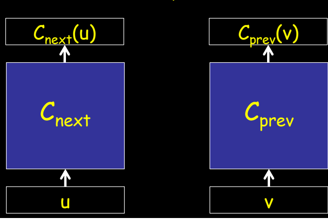
This is "PPAD" hard.

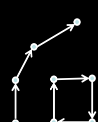
What's that?

What about just finding some NE?

Consider the following problem:

- Given two circuits C_{next} and C_{prev}, each with n-bit input, n-bit output.
- View as defining directed graph G: $u \rightarrow v$ iff $C_{next}(u)=v$ and $C_{prev}(v)=u$. (indeg ≤ 1, outdeg ≤ 1)





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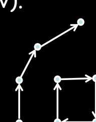
- View as defining directed graph G: $u \rightarrow v$ iff $C_{next}(u)=v$ and $C_{prev}(v)=u$. (indeg ≤ 1, outdeg ≤ 1)
- Say v "unbalanced" if indeg(v) $\neq$ outdeg(v).
- If 0^n is unbalanced, then find another unbalanced node. (must exist)



This is PPAD

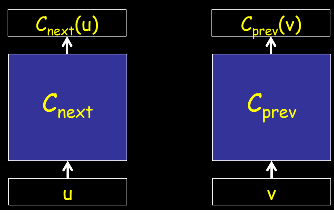
"END OF THE LINE"

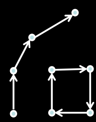
What about just finding some NE?

Why isn't this problem trivial? Say outdeg(0^n)=1.

- for(u = 0^n; u == $C_{prev}(C_{next}(u))$; u = $C_{next}(u)$);

Unfortunately, the path might be exponentially long.

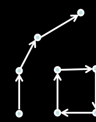




What about just finding some NE?

Not going to give proof that Nash is PPAD-hard.

Instead, give algorithm to show why Nash is in PPAD.



Lemke-Howson algorithm (1964)

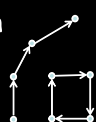
Preliminaries: [following discussion in Ch 2]

Given: matrices R,C.

- For simplicity, convert to symmetric game (A,A^T): $A = \begin{bmatrix} 0 & R \\ C^T & 0 \end{bmatrix}$

Claim: If ([x,y],[x,y]) is a symmetric equilib in (A,A^T), then (x/X,y/Y) is an equilib in (R,C).

Use $X = \sum_i x_i$, $Y = \sum_i y_i$



Pf: Each player getting payoff $x^TRy + y^TC^Tx$ with no incentive to deviate.

Lemke-Howson algorithm (1964)

Given nxn symmetric game A, find symm equil.

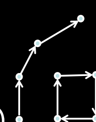
Consider the 2n linear constraints on n vars:

- $A_i z \leq 1$ for all i. ($A_i x \leq 1/Z$ where $x_i = z_i/Z$)
- $z_j \geq 0$ for all j. $z = (z_1, z_2, \ldots, z_n)$ If not zero…

Assume A is full rank, all A_{ij} non-neg.

- Implies have a bounded polytope.
- And all vertices have n tight constraints (at equality).

Alg will start at the origin (a vertex) and move along edges to a NE.



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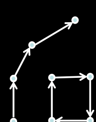
Strategy i is "represented" if $A_i z=1$ or $z_i=0$ (or both)

What if all strategies represented?

- Either z=(0,…,0) or (x,x) is a symmetric Nash.



Lemke-Howson algorithm (1964)

Alg: start at (0,…,0), move along edge.

(Relax one of z_j=0 and move until hit some $A_i z$=1)

- If i=j, then all strategies represented!
- Else i is represented twice.

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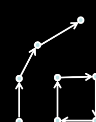

(0,…,0)

Lemke-Howson algorithm (1964)

Alg: start at (0,…,0), move along edge.

(Relax one of $z_j=0$ and move until hit some $A_i z=1$)

- If i=j, then all strategies represented!
- Else i is represented twice.

In general, take strategy represented twice and relax constraint you didn't just hit.

Claim: can't cycle or reach (0,…,0).

End is a Nash equilibrium.