CMU 15-896

Fair division: Complexity and approximation

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Complexity revisited

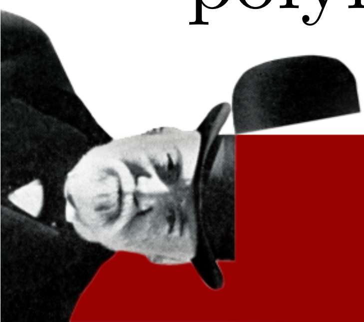
- Robertson-Webb model
  - $\text{Eval}_i(x, y) = V_i([x, y])$
  - $\text{Cut}_i(x, \alpha) = y$ s.t. $V_i([x, y]) = \alpha$
- Even-Paz is proportional and requires $O(n \log n)$ queries
- **Theorem [Edmonds and Pruhs, 2006]:** Any proportional protocol requires $\Omega(n \log n)$ queries
- We prove the theorem on the board

Approximate envy-freeness

- There is no known bounded envy-free (EF) protocol
- Can "efficiently" obtain ϵ-EF:
$$V_i(A_i) \geq V_i(A_j) - \epsilon$$
- Approach: ϵ-EF allocation of indivisible goods
- Setting: m goods, $V_i(S)$ denotes the value of agent $i \in N$ for the bundle S

Bounded EF

- Given allocation A, denote
$$e_{ij}(A) = \max\{0, V_i(A_j) - V_i(A_i)\}$$
$$e(A) = \max\{e_{ij}(A) : i, j \in N\}$$
- Define the maximum marginal utility
$$\alpha = \max\{V_i(S \cup \{x\}) - V_i(S) : i, S, x\}$$
- **Theorem [Lipton et al. 2004]:** An allocation with $e(A) \leq \alpha$ can be found in polynomial time

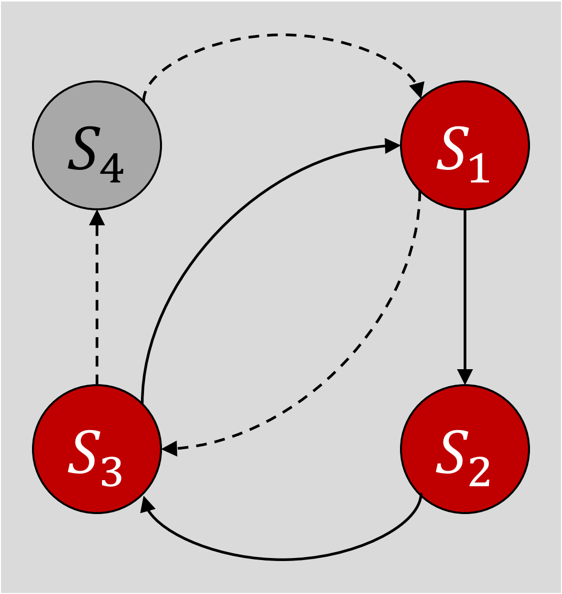


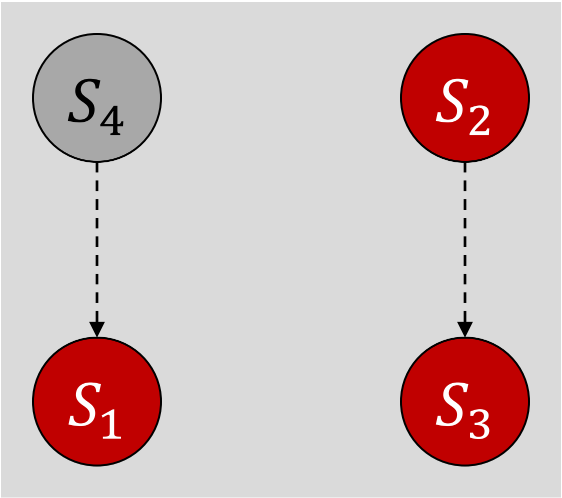
Proof of Theorem

- Given allocation A, we have an edge (i, j) in its envy graph if i envies j
- **Lemma**: Given partial allocation A with envy graph G, can find allocation B with acyclic envy graph H s.t. $e(B) \leq e(A)$

Proof of lemma

- If G has a cycle C, shift allocations along C to obtain A'; clearly $e(A') \leq e(A)$
- #edges in envy graph of A' decreased:
  - Same edges between $N \setminus C$
  - Edges from $N \setminus C$ to C shifted
  - Edges from C to $N \setminus C$ can only decrease
  - Edges inside C decreased
- Iteratively remove cycles ■





Proof of Theorem

- Maintain envy $\leq \alpha$ and acyclic graph
- In round 1, allocate good g_1 to arbitrary agent
- $g_1, \ldots, g_{k-1}$ are allocated in acyclic A
- Derive B by allocating g_k to source i
- $e_{ji}(B) \leq e_{ji}(A) + \alpha = \alpha$
- Use lemma to eliminate cycles ■

Back to cakes

- Agent i makes $1/\epsilon$ marks $x_1^i, \ldots, x_{1/\epsilon}^i$ such that for every k, $V_i\left(\left[x_k^i, x_{k+1}^i\right]\right) = \epsilon$
- If intervals between consecutive marks are indivisible goods then $\alpha \leq \epsilon$
- Now we can apply the theorem
- Need n/ϵ cut queries and n^2/ϵ eval queries

An even simpler solution

- Relies on additive valuations
- Create the "indivisible goods" like before
- Agents choose pieces in a round-robin fashion: $1, \dots, n, 1, \dots, n, \dots$
- Each good chosen by agent i is preferred to the next good chosen by agent j
- This may not account for the first good g chosen by j, but $V_i(\{g\}) \le \epsilon$