Cake cutting

- A cake must be divided between several children
- The cake is heterogeneous
- Each child has different value for same piece of cake
- How can we divide the cake fairly?
- What is “fairly”?  
- A metaphor for land disputes, time using shared resources, etc.
The model

• Cake is interval $[0,1]$
• Set of agents/players $N = \{1, \ldots, n\}$
• Piece of cake $X \subseteq [0,1]$: finite union of disjoint intervals
• Each agent has valuation $V_i$ over pieces of cake
  o Additive: for $X \cap Y = \emptyset$, $V_i(X) + V_i(Y) = V_i(X \cup Y)$
  o For all $i \in N$, $V_i([0,1]) = 1$
  o Divisible: $\forall \lambda \in [0,1]$ can cut $I' \subseteq I$ s.t. $V_i(I') = \lambda V_i(I)$
• Find allocation $A_1, \ldots, A_n$
  o Not necessarily connected pieces
Fairness properties

• Proportionality:
  \[ \forall i \in N, V_i(A_i) \geq \frac{1}{n} \]

• Envy-Freeness (EF):
  \[ \forall i, j \in N, V_i(A_i) \geq V_i(A_j) \]

• Vote: For \( n = 2 \) which is stronger?
• Vote: For \( n \geq 3 \) which is stronger?
Cut-and-Choose

• Algorithm for $n = 2$
• Agent 1 divides into two pieces $X, Y$ s.t.
  \[ V_1(X) = 1/2, V_1(Y) = 1/2 \]
• Agent 2 chooses preferred piece
• This is EF (hence proportional)
The Robertson-Webb model

- A concrete complexity model
- Two types of queries
  - \( \text{Eval}_i(x, y) = V_i([x, y]) \)
  - \( \text{Cut}_i(x, \alpha) = y \text{ s.t. } V_i([x, y]) = \alpha \)
- Vote: Minimum \#queries needed to find an EF allocation when \( n = 2 \)?
Dubins-Spanier

• Referee continuously moves knife
• Repeat: when piece left of knife is worth $1/n$ to agent, agent shouts “stop” and gets piece
• That agent is removed
• Last agent gets remaining piece
• Protocol is proportional
Discrete Dubins-Spanier

• Moving knife is not really needed
• Repeat: each agent makes a mark at his $1/n$ point, leftmost agent gets piece up to its mark
• The protocol is proportional
Example

1/3
Example

\[ \frac{1}{3} \quad \frac{1}{3} \]

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Example

\[ \frac{1}{3} \quad \frac{1}{3} \quad \frac{1}{2} \]
Even-Paz

• Given \([x, y]\), assume \(n = 2^k\)

• Each agent \(i\) makes a mark \(z\) such that

\[
V_i([x, z]) = \frac{1}{2} V_i([x, y])
\]

• Let \(z^*\) be the \(n/2\) mark from the left

• Recurse on \([x, z^*]\) with the left \(n/2\) agents, and on \([z^*, y]\) with the right \(n/2\) agents

• The protocol is proportional
Complexity of proportionality

• Dubins-Spanier requires $\Theta(n^2)$ queries in the RW model

• Even-Paz requires $\Theta(n \log n)$ queries in the RW model

• Theorem [Edmonds and Pruhs, 2006]: Any proportional protocol needs $\Omega(n \log n)$

[We’ll prove on Tuesday]
Selfridge-Conway

- **Stage 0**
  - Agent 1 divides the cake into three equal pieces according to $V_1$
  - Agent 2 trims the largest piece s.t. there is a tie between the two largest pieces according to $V_2$
  - Cake 1 = cake w/o trimmings, Cake 2 = trimmings

- **Stage 1 (division of Cake 1)**
  - Agent 3 chooses one of the three pieces of Cake 1
  - If agent 3 did not choose the trimmed piece, agent 2 is allocated the trimmed piece
  - Otherwise, agent 2 chooses one of the two remaining pieces
  - Agent 1 gets the remaining piece
  - Denote the agent $i \in \{2, 3\}$ that received the trimmed piece by $T$, and the other by $T'$

- **Stage 2 (division of Cake 2)**
  - $T'$ divides Cake 2 into three equal pieces according to $V_{T'}$
  - Agents $T$, 1, and $T'$ choose the pieces of Cake 2, in that order
**RW is for honest kids**

- EF protocol that uses $n$ queries
- $f = 1$-1 mapping from valuation functions to $[0,1]$
- The protocol asks each agent $\text{cut}_i(0, 1/2)$
- Agent $i$ replies with $y_i = f(V_i)$
- The protocol computes $V_i = f^{-1}(y_i)$
- We therefore need to assume that agents are “honest”
Complexity of EF

- $n = 2$: Cut and Choose
- $n = 3$: “good” protocol [Selfridge and Conway]
- $n \geq 4$: known protocol requires unbounded #queries [Brams and Taylor, 1995]
- Lower bound of $\Omega(n^2)$ [P, 2009], unbounded with contiguous pieces [Stromquist, 2009]
Price of fairness

- Social welfare of $A = \sum_{i \in N} V_i(A_i)$
- Requires interpersonal comparison of utils
- Price of EF = worst-case (over valuation functions) ratio between social welfare of the best allocation and social welfare of the best EF allocation
- Theorem [Caragiannis et al. 2009]: The price of EF is $\Omega(\sqrt{n})$
Proof of Theorem

- Agents 1, ..., $\sqrt{n}$ uniformly desire disjoint intervals of length $1/\sqrt{n}$
- The others uniformly desire the whole cake
- Optimal solution: give whole cake to the “focused” agent $\Rightarrow$ SW = $\sqrt{n}$
- Any EF solution must give $\frac{n-\sqrt{n}}{n}$-fraction to the “unfocused” agents $\Rightarrow$ SW $\leq$ 2  ■
The dumping paradox

• If connected pieces must be allocated, by throwing away pieces, can increase the welfare of optimal EF allocation by a factor of $\sqrt{n}$ [Arzi et al. 2011]
• Example: for $n = 2$, can increase from 1 to $\sim 3/2$