CMU 15-896

TEACHERS:
AVRIM BLUM
ARIEL PROCACCIA (THIS TIME)

SOCIAL CHOICE:
THE AXIOMATIC APPROACH
Axiomatic approach

• Social choice theory often uses axioms to guide the design of voting rules
• Representation theorem: a set of axioms that uniquely characterize a popular rule
• This approach has been applied to ranking systems, collaborative filtering, recommendation systems, etc.
• Coming up: representation theorem for PageRank
The page ranking problem

- The internet is represented by a directed graph $G = (V, E)$
- Vertices $V$ are webpages
- $(u, v) \in E$ represents a hyperlink from $u$ to $v$
- Given $G$, a ranking system produces a ranking over $V$ that represents the “power” or “relevance” of webpages
- From a social choice point of view, the sets of voters and alternatives coincide
**PageRank**

- Rank the vertices based on the stationary probability of a random walk on the graph
- Assume that the graph is strongly connected
- Define the matrix \( A_G \)

\[
[A_G]_{ij} = \begin{cases} 
  \frac{1}{|S(v_j)|} & (v_j, v_i) \in E \\
  0 & \text{Otherwise}
\end{cases}
\]
PageRank

• The PageRank of $G$ is $r$ such that
  \[ A_G r = r \]
  
• The PageRank ranking system ranks $V$ according to $r$:
  \[ v_i \geq_{PR} v_j \iff r_i \geq r_j \]
Axiom 1: isomorphism

- The ranking must not rely on the names of the vertices, only on the voting structure.
- Clearly satisfied by PageRank.
Axiom 2: Vote by Committee

- A node may vote indirectly through intermediate nodes, each of which has the original votes

![Diagram showing the concept of vote by committee with nodes a, b, and c, and their connections.]
**Vote by committee formalized**

- Ranking system $f$ satisfies *vote by committee* if for every $G = (V, E)$, for every $v, v', v'' \in V$, and for every $k \in \mathbb{N}$, if $G' = (V', E')$ where
  \[ V' = V \cup \{u_1, \ldots, u_k\} \]
  and
  \[ E' = E \setminus \{(v, x)|x \in S_G(v)\} \cup \{(v, u_i)|i = 1, \ldots, k\} \]
  \[ \cup \{(u_i, x)|x \in S_G(v), i = 1, \ldots, k\}, \]
  then $v' \preceq^f_G v'' \iff v' \preceq^f_G v''$

- **Lemma:** PageRank satisfies vote by committee
Proof

- Let \( \mathbf{r} \) be a solution to \( A_G \mathbf{r} = \mathbf{r} \)

- \( \mathbf{r}' = \left( r_1, \ldots, r_n, \frac{r_1}{k}, \ldots, \frac{r_1}{k} \right)^T \)

- \( A'_G = \begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} & a_{11} & \cdots & a_{11} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} & a_{n1} & \cdots & a_{n1} \\ \frac{1}{k} & \vdots & \vdots & 0 & \vdots & \vdots & \vdots \\ \frac{1}{k} & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{k} & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \)

- For \( i = 1, \ldots, n \): \( [A'_G \mathbf{r}']_i = \sum_{j=2}^n a_{ij} r_j + k a_{i1} \cdot \frac{r_1}{k} = r_i \)

- For \( i = n + 1, \ldots, n + k \): \( [A'_G \mathbf{r}']_i = \frac{1}{k} r_1 \)
Axiom 3: self edge

- Adding a self edge to $v$ strengthens $v$ but does not change the ranking of other vertices
**Axiom 4: collapsing**

- Vertices that vote identically can be merged into a single vertex, with all the incoming edges of the original vertices.
- The ranking of vertices that were not collapsed remains unchanged.

Diagram:
- Original graph with vertices $a$ and $b$.
- Merged vertex $a$ with all incoming edges from the original vertices.
**Axiom 5: proxy**

- $k$ vertices of equal rank that voted for $k$ alternatives via proxy can achieve the same result by voting for one alternative each.
Representation theorem

• Theorem [Altman and Tennenholtz 2005]: a ranking system satisfies axioms 1-5 if and only if it is the PageRank ranking system

• To show “only if”: prove that the five axioms imply a unique ranking on each graph!
Selecting a subset

• A $k$-selection system receives a directed graph as input and outputs $V' \subseteq V$ such that $|V'| = k$
• Edges are interpreted as approval votes, trust, or support
• Think of graph as directed social network
• A $k$-selection system $f$ is impartial if $i \in f(G)$ does not depend on the votes of $i$
Impartial approximations

- Optimization target: sum of indegrees of selected agents
- Optimal solution: not impartial
- $k = n$: no problem
- $k = 1$: no finite impartial approx
- $k = n - 1$: no finite impartial approx!
An impossibility result

• Theorem [Alon et al. 2011]: For all $k \in \{1, \ldots, n - 1\}$ there is no impartial $k$-selection system w. finite approx ratio

• Proof ($k = n - 1$):
  - Assume for contradiction
  - Wlog $n$ eliminated given empty graph
  - Consider stars with $n$ as center, $n$ cannot be eliminated
  - Function $f: \{0,1\}^{n-1} \setminus \{\vec{0}\} \rightarrow \{1, \ldots, n-1\}$ satisfies $f(\vec{x}) = i \iff f(\vec{x} + e_i) = i$
  - $|f^{-1}(i)|$ even for all $i = 1, \ldots, n - 1 \Rightarrow |\text{dom}(f)|$ is even; but $|\text{dom}(f)| = 2^{n-1} - 1$
A mathematician’s survivor

• Each tribe member votes for at most one member
• One member must be eliminated
• Impartial rule cannot have property: if unique member received votes he is not eliminated
Randomized systems

• The randomized $m$-partition system:
  o Assign vertices uniformly i.i.d. to $m$ subsets
  o For each subset, select $\sim \frac{k}{m}$ agents with highest indegrees based on edges from other subsets
• The $m$-partition system is a distribution over impartial systems
Approximation

- Theorem [Alon et al. 2011]:
  1. The approx ratio is 4 with $m = 2$
  2. The approx ratio is $1 + O\left(\frac{1}{k^3}\right)$ for $m \sim k^{\frac{1}{3}}$

- Proof (only part 1):
  - Assume for ease of exposition: $k$ is even
  - Let $K$ be the optimal set
  - A partition $\pi = (\pi_1, \pi_2)$ divides $K$ into two subsets $K_{1\pi} = K \cap \pi_1$ and $K_{2\pi} = K \cap \pi_2$
  - $d_{1\pi} = \{(u, v) \in E | u \in \pi_2, v \in K_{1\pi}\}$, $d_{2\pi}$ defined analogously
  - We get at least $\frac{d_{1\pi} + d_{2\pi}}{2}$
  - $\mathbb{E}[d_{1\pi} + d_{2\pi}] = \frac{OPT}{2} \blacksquare$